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ON THE THEORY OF UNSTEADY PLANING AND THE MOTION OF A WING WITH VORTEX SEPARATION

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I. PRELIMINARY OBSERVATIONS

The disturbances imparted to the water by a planing body give rise to a wave form of motion on the free surface, the length of the waves increasing indefinitely with increase in the Froude number and being directly proportional to the latter in the case of the plane or two-dimensional problem. Near the planing surface the general picture of the flow as shown by tests presents a true jet or spray character; i.e., at some distance ahead of the body the water surface is practically undisturbed, while immediately forward of the body the water is thrown off in a spray.

The high-speed planing motion of the body gives rise to very large accelerations in the fluid and, in this respect, resembles the phenomenon of impact. The chief forces that determine the motion of the particles of fluid near the body appear to be the result of the large pressure gradients. As in the case of impact, it is therefore permissible to neglect the weight of the water.

The dynamic reaction of the water is completely determined by its motion in the immediate neighborhood of the planing body. At large Froude numbers the effect of the weight shows up to any appreciable extent only at some distance from the body, so that the flow near the body can be considered as part of a flow of an infinitely extending weightless fluid. The same conclusion can also be reached from another point of view. Let us consider a series of motions for which the angles of inclination to the water surface are the same and the wetted portions of the bottoms geometrically similar. Applying the Lagrange integral to the absolute potential motion of the heavy fluid, we obtain the boundary condition of constant pressure at the free surface in the form

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$$\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + gy = 0 \tag{1}$$

where t is the time,

φ the velocity potential,

v the velocity of the fluid,

g the acceleration of gravity,

and y a coordinate taken normal to the initial water level.

For steady planing, we have

$$\frac{\partial\varphi}{\partial t} = -c\,\frac{\partial\varphi}{\partial x}$$

where c is the translational velocity in the forward direction coinciding with the x axis. We shall introduce nondimensional magnitudes with the aid of the relations:

$$\frac{\partial\varphi_1}{\partial x_1} = c\,\frac{\partial\varphi_1}{\partial x_1};\ v = cv_1;\ y = hy_1$$

where h is a certain characteristic dimension. Boundary condition (1) then assumes the form

$$-\frac{\partial\varphi_1}{\partial x_1} + \frac{{v_1}^2}{2} + \frac{gh}{c^2}\ y_1 = 0 \tag{2}$$

From relation (2), it is clear that for the same values of the Froude number $F = \frac{c^2}{gh}$ the motions become dynamically similar. The values $\frac{\partial\varphi_1}{\partial x_1}$, v_1, and y_1 at the free surface everywhere except at a very small region at the edges of the planing surface are of the order of the angle of attack β, which will be assumed as infinitely small in what follows. If the motion of the fluid be determined by the methods of the theory of waves of small amplitude, then in condition (2) it is necessary to neglect ${v_1}^2$. At a small value of the Froude number $F = \frac{c^2}{gh}$ the first and last terms in (2) are of the order of β and in this case the weight cannot be neglected. For a large value of the Froude number, however, $F \approx \frac{1}{\beta}$, the third term has the order of β^2, and therefore to an accuracy of the second order of smallness.

$$\frac{\partial \varphi_1}{\partial x_1} = 0 \tag{3}$$

Thus, for large values of the Froude number F, in the approximate solution, it is necessary to use boundary condition (3), which is equivalent to assuming the fluid as weightless. In what follows, the case will be considered where the planing is assumed to take place at high Froude numbers.

It will be shown below (section II) that in the above case the motion of the water at each instant of time may be considered as the result of the simultaneous action on it of a system of impulses distributed over the area swept out by the planing surface, the variation with time of the impact pressures on the free surface being neglected. Thus stated, the planing problem differs from that of impact on the water only in that in the former it is necessary to take account of the disturbance of the water remaining behind or, expressed otherwise, it is necessary to take account of the asymmetry of the flow in front of and behind the body.

The energy required to maintain the motion of the body may be considered as consisting of the energy of the disturbed water and the energy dissipated by the dissipative forces of viscosity. The drag may be computed by the energy imparted to the fluid by the body in a unit of distance. The energy of the disturbance is made up of the energy contained in the sprays thrown off ahead of the body and that of the disturbances remaining behind.

The flow phenomena at the edges in the case of the two-dimensional problem were made the subject of special study by Wagner (reference 1). In his work, he presents methods for the theoretical computation of the portion of the drag contributed by the spray at the forward edge. That portion of the drag due to the disturbance remaining behind the body is analogous to the induced drag of a wing.

For establishing the initial disturbance of the main flow, the effect of the viscosity, by comparison with that of the inertia of the water, is negligible. On the other hand, the viscosity is of appreciable effect on the spray motion of the water and on the boundary-layer motion. At small angles of attack, the drag due to the fluid friction at the bottom may be very large. The computation of this drag is complicated by the fact that at the forward portion the frictional forces on the bottom are directed forward, so that the speed of the water in the spray is greater than that of the bottom.

The exact solution of the three-dimensional hydrodynamic problem of the planing motion of a body on the surface of an incompressible, ideal, and weightless fluid presents insurmountable mathematical difficulties and may therefore be treated only approximately. The two-dimensional problem of the steady planing motion belongs to the type of flow problems considered by Kirchhoff. The solution of the two-dimensional planing problem for a flat plate has been given by Chaplygin with the participation of Gurevitch and Yanpolsky (reference 2). A comparison of this solution with the results of tests shows a very good qualitative agreement. Particularly noteworthy is the good qualitative agreement of the theoretical law of pressure distribution with the pressure distribution determined by experiment. The lack of quantitative agreement may be explained by the finite span employed in the tests since the pressure strongly depends on the span.

In a fundamental paper on the theory of planing, Wagner, investigating on the one hand flows of the type of Kirchhoff and on the other rotational flows about thin profiles in an infinite fluid, showed that for infinitesimal angles of attack, to an accuracy of the second order of smallness, the lift force on the planing surface is equal to half the lift on a wing of the same profile. The flows at the edges of the wing and the planing surface are different in character. In particular, in planing, thin sprays are obtained at the forward edge, which result in an additional drag, whereas, in the case of a wing, there are suction forces at the corresponding positions. According to Wagner, the drag due to the formation of the sprays is equal in magnitude to half the suction force on the corresponding wing. It is for this reason that the drag due to the sprays is not difficult to compute. The conclusion also readily follows that the wing always has better characteristics than the corresponding planing surface. From the mathematical point of view, the planing problem can easily be reduced, as we shall show below, to that of the wing theory by making use of the approximate methods that have been applied with great success in the theory of thin wings and in the theory of waves of small amplitude. The approximations made are equivalent to those of Wagner.

On figure 1 are given the experimental data derived from the tests of Sottorf (reference 3). The tests were conducted on flat plates. The aspect ratios* $\lambda = l'/b$ where l' is the wetted length, as measured in the tests and b the span, laid off on the x axis. On the ordinate axis the coefficient K is laid off:

$$K = \frac{A}{\rho b l' c^2 \beta} \qquad (4)$$

where A is the lift force

and ρ the density of the water.

From these data, corresponding to various conditions of motion, it may be seen that the coefficient K depends only on the ratio $\frac{l'}{b}$. The curve of $K = \frac{\pi}{2(1 + 2\lambda)}$ indicated by the solid line shows the dependence on the aspect ratio of the value of the coefficient for half the lift force of the wing as given by the theory of the lifting line for elliptic distribution of the circulation. Comparing this curve with the test points, we may observe a qualitative agreement. For $\lambda \sim 2$, we also have a good quantitative agreement. It should be observed that the experimental values of the coefficient K for the wings also do not entirely coincide with the theoretical values and for small values of λ, for example, they differ as in the given case.

Proceeding to the study of the unsteady or nonuniform motion of the body within or along the surface of the fluid

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*In aerodynamics, the aspect ratio has been defined as b/l. When applied to planing, it seemed to us more convenient to take as the aspect ratio the ratio l'/b, since the terms l' the wetted length and b the width of step, may be considered as definitely established parameters.

we are immediately confonted with the difficulties characteristic of these problems. In the case of unsteady motion, the mechanical characteristics of the body-fluid system are not in general defined by very simple geometric and kinematic data for the body at the instant considered as is the case for steady motion. If, for example, in a given time interval, we have a constant accelerated forward motion, it would be incorrect to say that the hydrodynamic reaction of the water may be expressed as a function of the velocity, acceleration, and geometric parameters giving the body position. The reaction of the water will, in general, also depend on the character of motion of the body in the preceding time interval, since the body may have previously given a large disturbance to the water.

In the theoretical study of unsteady motion, a number of assumptions are usually made for computing the hydrodynamic forces. One most often made is the so-called "stationary forces" hypothesis. According to the latter, it is assumed that during the unsteady motion the forces at each instant coincide with those of a corresponding steady motion defined by the same geometric parameters giving the position of the body and its translational and angular velocities. The values of the forces determined by the stationary forces hypothesis in some instances differ considerably from their actual values in the unsteady motion.

Our central problem is to study the relations between the forces obtained according to the stationary hypothesis and other assumptions and the actual values of these forces. Our work constitutes an application to the planing problem of the methods and results of the theory of unsteady flow about a wing. In addition, we shall attempt to generalize and simplify the methods considrerd and present them in a form where they may be conveniently used for study and for practical applications.

II. FORMULATION OF THE PLANING PROBLEM AND ITS RELATION TO THE PROBLEM OF A THIN WING

We shall consider the main part of the flow, where the water moves continuously. The flow will possess an infinite velocity at the edges of the planing surface, where a thin spray is thrown off.

We shall determine the motion of the fluid on the basis of the following assumptions:

1. As shown by experiment, there is no spray separating at the trailing edge at large planing speeds. The water, in this case, flows off smoothly tangent to the bottom. We shall correspondingly require that the velocity of the fluid at the trailing edge be finite.

2. Analogous to what is usually assumed in the theory of a thin wing when formulating the flow conditions and in the theory of waves of small amplitude, the boundary conditions for the determination of the potential flow on the surface of the water and at the bottom, we shall transfer along a vertical to the horizontal surface coinciding with the undisturbed water level. We shall thus reduce the problem to that of the determination of the potential flow of the fluid at the lower portion of the half space bounded by the horizontal plane. This approximation clearly does not hold true in the region of formation of the spray and at the spray itself. The motion of the water in the spray does not, however, have any appreciable effect on the disturbances of the main mass of fluid by which disturbances, the chief terms of the water reaction, are determined. Without too great an error, it is permissible to neglect the spray. It may be shown that at small angles of attack β the thickness of the spray and the momentum of the water thrown off in the spray in a unit of time are of the order of β^2 (reference 1). In what follows, we shall assume that β is very small and shall neglect small quantities of the order of β^2 (references 4 and 5).

3. In the dynamical boundary condition on the free surface, we shall neglect the weight of the water and the squares of the absolute velocities of the water. It may be shown that on the free surface everywhere, with the exception of the small region at the edges where the spray is formed, the magnitude of the absolute velocity v is of the order of β and therefore $v^2 \sim \beta^2$. The assumption of large planing velocities in the horizontal direction justifies the neglect of the weight of the water.

We shall see below that the assumptions enumerated above immediately reduce the planing problem to that of the motion of a wing. For a steady planing motion, as we have seen in the preceding paragraph, such a description corresponds to actual physical laws. For the unsteady motion, this method of formulating the problem when applied to the wing leads to theoretical conclusions that show good agreement with test results.

With the aid of the assumptions enumerated above, it is not difficult to formulate mathematically the boundary conditions for the determination of the unsteady motion of the fluid.

Let us consider the two-dimensional problem. In the plane of motion the wetted portion of the bottom profile is represented by the curved segment M'N' (fig. 2). We shall study the case where the curve M'N' differs slightly, in the sense indicated below, from its projection MN on the undisturbed surface.

Let some point O_1 on the initial water level be the origin of a fixed system of coordinates. The point O, the center of MN, we shall take as the origin of the movable system of coordinates Oxy, the y axis being directed vertically upward and the x axis horizontal.

We shall denote by $\varphi(x, y, t)$ the velocity potential of the absolute motion of the fluid. Referring the boundary conditions to the x axis, we shall have along MN the condition

$$\frac{\partial\varphi}{\partial y} = - v_n \qquad (1)$$

where v_n is the normal component of the velocity of the bottom. The dependence of v_n on x and on the time t is determined by the geometric and kinematic characteristics of the planing surface. Let $y = f(x)$ be the equation of the curve M'N' for some position of the bottom. We shall limit ourselves to the case where $f(x)$ and $f'(x)$ are small. After an infinitesimal displacement of the bottom, we shall have for dy/dx, to an accuracy of the second order of smallness:

$$\frac{dy}{dx} = \beta + f'(x)$$

where β is the angle of rotation. If c and v are, respectively, the horizontal and vertical components of the velocity of point M' and ω the angular velocity of the bottom, we may write for v_n

$$v_n = -v - \omega(a + x) + c\frac{dy}{dx}$$

or

$$v_n = -(v + \omega a) - \omega x + c\beta + cf'(x) = v_1 - \omega x + cf'(x) \qquad (2)$$

where

$$v_1 = -(v + \omega a) + c\beta$$

For a plate forming an angle β with the horizontal, we obtain

$$v_n = v_1 - \omega x \qquad (2a)$$

If the profile M'N' undergoes bending, then f' is a function not only of x but also of the time t. In this case, on the right-hand side of formula (2), it is necessary to add the normal component of the velocity due to the motion of deformation.

Since the fluid is at rest at infinity, the Lagrange integral on the free surface gives:

$$\frac{\partial\varphi(\alpha,t)}{\partial t} + \frac{1}{2}v^2 + gy = 0$$

where g is the acceleration of gravity, and α is a coordinate in the fixed system of axes (fig. 2). Discarding the terms gy and $\frac{1}{2}v^2$ in accordance with assumption 3, we obtain

$$\frac{\partial\varphi(\alpha,t)}{\partial t} = 0 \qquad (4)$$

Thus, on the free surface φ maintains a constant value with respect to time although it may differ in value from point to point. We shall assume that the motion began from the state of rest and considering the initial value of φ to be zero, we have everywhere on the free surface, over which the body has not passed, the condition

$$\varphi = 0 \qquad (4a)$$

We shall denote by S the path of the planing body on the water surface consisting of the free portion and the portion in contact with the bottom at any given instant. On the free portion of the surface S, the function $\varphi(\alpha)$ is determined by the character of the planing and may be found from the condition that the velocity of the fluid at M is finite.

The potential of the unsteady motion of the fluid at each given instant may be considered as arising from the impact of a system of impulses $p_t(\alpha) = -\rho\varphi(\alpha)$ distributed over the surfaces. At a succeeding time instant, this surface will be $S' > S$ and the system of impulses will be p_t' where, on the free portion of the surface S, $p_t' = p_t$.

The potential function $w(z) = \varphi + i\psi$ may be extended to the upper half plane on the basis of the Schwarz principle of symmetry since to the right of N on the Ox axis, $\varphi = 0$. Let B (fig. 2) be the position of the point M when the unsteady motion is set up. We then obtain the extended function $w(z)$, which is holomorphic throughout the plane BN or AN (the point A denotes $-\infty$) if the circulation about an infinitely removed contour is different from zero, and

$$\varphi(x,y) = -\varphi\,(x - y)$$

$$\psi(x,y) = \psi(x - y)$$

Therefore, at the separating edges above and below, the value of φ is at each instant the same in magnitude but opposite in sign:

$$\varphi(C) = -\varphi(C_1)$$

It therefore follows that for the motion of an infinite fluid determined by the function $w(z)$ the line BN is a line of discontinuity of the horizontal component of the velocity of the fluid. The vertical component of the velocity changes continuously in passing through BN. The horizontal components above and below are equal in magnitude but oppositely directed. Denoting the discontinuity in the velocity by $\gamma(\alpha)$, we have

$$\gamma(\alpha) = \frac{d\varphi}{d\alpha}\,(C_1) - \frac{d\varphi}{d\alpha}\,(C) = 2\,\frac{d\varphi}{d\alpha}\,(C_1) \qquad (5)$$

and $\gamma(\alpha)$ may be considered as the density of the vortices distributed along BM.

Let $\Gamma(\alpha)$ denote the circulation taken about some closed path L (fig. 2) cutting AN at point O in the direction shown on the sketch.

Obviously

$$\Gamma(\alpha) = -2\varphi(O_1)$$

whence

$$\frac{d\Gamma}{d\alpha} = -\gamma(\alpha) \tag{6}$$

$\Gamma(\alpha_1)$ is the circulation about MN at the instant considered. If the circulation about MN is known as a function of time, then, considering the various points on AM as the positions of point M at various instants of the motion

$$\alpha = \alpha_0 + \int_{t_0} c dt \tag{7}$$

and we may compute $\gamma(\alpha)$ by the formula

$$\gamma(\alpha) = -\frac{d\Gamma}{dt}\frac{dt}{d\alpha} = -\frac{1}{c}\frac{d\Gamma}{dt} \tag{8}$$

Conditions (1), (4), and (4a) and also the extension of the flow in the upper half region apply in the same form also to the three-dimensional problem with the difference only that the discontinuity of the horizontal component of the velocity behind the planing body depends in this case not only on the longitudinal coordinate, but also on the coordinate in the transverse direction.

The wetted length $M'N' \sim MN = 2a$ in unsteady planing may change with time. The wetted length as a function of the time depends very much on the form and dimensions of the bottom, on the amount of immersion, the angle of attack, and also on the weight of the water. It is important to note that the wetted length cannot be given as a parameter of the motion but is determined in the process of solution of the hydrodynamical problem.

If a is considered as constant, then the problem we have formulated coincides exactly with that of the determination of the unsteady motion of an infinite fluid about the thin profile M'N' with a surface of discontinuity separating at the trailing edge. The problem of the motion of a flat plate with a line of discontinuity has thus been formulated by Wagner (reference 6) and that of the oscillations set up has been considered by Glauert (reference 7) and Keldysh and Lavrentiev (reference 8). We shall generalize this problem, extending it to the case of a variable a and a thin profile of any form. Moreover, we shall investigate in greater detail the physical character of the hydrodynamic forces and shall study the forces in the unsteady planing motion.

III. DETERMINATION OF THE FLUID FLOW

To determine the potential function $w(z,t)$ of the fluid, we have, on the basis of the assumptions made, the following conditions (for notation, see fig. 2):

1) Outside the segment BN

$$\frac{dw}{dz} = u - iv$$

is a holomorphic and single-valued function of z.

2) At infinity, the fluid is at rest but the circulation about a contour L_2 enveloping BN is in general different from zero and is equal to Γ_0 = const. Expressed otherwise, the series development of dw/dz near an infinitely removed point has the following form:

$$\frac{dw}{dz} = \frac{\Gamma_0}{2\pi i}\frac{1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots$$

3) Along MN on both sides

$$\frac{\partial\varphi}{\partial y} = -v_n$$

and $\frac{\partial\varphi}{\partial y}$ is continuous along BM.

4) Along BM, we have a discontinuity in the horizontal component of the velocity:

$$\gamma(\alpha) = \frac{d\varphi}{d\alpha}(C_1) - \frac{d\varphi}{d\alpha}(C)$$

The function $\gamma(\alpha)$ should be determined so that at point M the velocity of the fluid is always finite.

Let us consider the function

$$F(z) = \sqrt{z^2 - a^2}\,\frac{dw}{dz}$$

Let us take the branch of $\sqrt{z^2 - a^2}$, which for $x > a > 0$ gives $+\sqrt{x^2 - a^2}$. Then in approaching MN from above

$$\sqrt{z^2 - a^2} \to + i\,\sqrt{a^2 - x^2}$$

and in approaching MN from below

$$\sqrt{z^2 - a^2} \to - i\,\sqrt{a^2 - x^2}$$

Along AM we have $-\sqrt{|x^2 - a^2|}$.

The function F(z) is holomorphic and single-valued everywhere outside the segment BN. Near an infinitely removed point the series below holds true, namely:

$$F(z) = \frac{\Gamma_0}{2\pi i} + \frac{c_2}{z} + \ldots$$

Applying the formula of Cauchy to the function F(z), we may write

$$F(z) = -\frac{1}{2\pi i}\int_{L_1}\frac{F(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{L_2}\frac{F(\zeta)d\zeta}{\zeta - z}$$

where the integration for both integrals is to be performed in the counterclockwise direction.

Contracting the path L_1 to BN and expanding path L_2 to infinity, we obtain from the above expansion for F(z) at infinity

$$F(z) = \frac{1}{2\pi i}\int_B^N \frac{F(x_0 + i0) - F(x_0 - i0)}{x_0 - z}\,dx_0 + \frac{\Gamma_0}{2\pi i}$$

Denoting by $u_1 - iv_1$ and $u_2 - iv_2$ the values of dw/dz in approaching BN from above and below, we have

$$u_2 = -u_1$$

and along MN

$$v_2 = v_1 = -v_n$$

We thus find along MN

$$F(x_0 + i0) - F(x_0 - i0) = -2v_n(x_0)\sqrt{a^2 - x_0^2}$$

and along BM

$$F(x_0 + i0) - F(x_0 - i0) = 2u_1\sqrt{x_0^2 - a^2} = \gamma(\alpha)\sqrt{x_0^2 - a^2}$$

Hence:

$$\frac{dw}{dz} = -\frac{1}{2\pi i\sqrt{z^2 - a^2}}\int_{-a}^{+a}\frac{2v_n\sqrt{a^2 - x_0^2}}{x_0 - z}\,dx_0 + $$

$$+\frac{1}{2\pi i\sqrt{z^2 - a^2}}\int_{\alpha_0}^{\alpha_1}\frac{\gamma(\alpha)\sqrt{x_0^2 - a^2}}{x_0 - z}\,d\alpha + \frac{\Gamma_0}{2\pi i\sqrt{z^2 - a^2}} \tag{1}$$

Between α and x_0, there is the relation

$$x_0 = \alpha - \alpha_1 - a$$

The first term of formula (1) gives the velocity field of the irrotational motion of the fluid at a given distribution of the normal velocities along MN. The second term gives the velocity field due to the vortices springing from the trailing edge of the plate and distributed along BM, in the presence of the fixed plate MN, with density $\gamma(\alpha)$. Finally, the third term gives the pure motion of circulation about the fixed plate.

In equation (1) replacing v_n by $v_1 - \omega x_0 + cf'(x_0)$ in accordance with formula (2), section II, and performing the integration of the first integral, we obtain

$$\frac{dw}{dz} = iv_1\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right) + \frac{i\omega}{2}\,\frac{(z - \sqrt{z^2 - a^2})^2}{\sqrt{z^2 - a^2}}$$

$$+ \frac{c}{\pi i\sqrt{z^2 - a^2}} \int_{-a}^{+a} \frac{f'(x_0)\sqrt{a^2 - x_0^2}}{z - x_0}\, dx_0$$

$$+ \frac{1}{2\pi i\sqrt{z^2 - a^2}} \int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\sqrt{x_0^2 - a^2}}{x_0 - z}\, d\alpha + \frac{\Gamma_0}{2\pi i\sqrt{z^2 - a^2}} \qquad (2)$$

whence

$$w = iv_1(z - \sqrt{z^2 - a^2}) - \frac{i\omega}{4}(z - \sqrt{z^2 - a^2})^2 +$$

$$+ \frac{c}{\pi} \int_{-a}^{+a} f'(x_0) \ln \frac{x_0 - i\sqrt{a^2 - x_0^2} - z - \sqrt{z^2 - a^2}}{x_0 + i\sqrt{a^2 - x_0^2} - z - \sqrt{z^2 - a^2}}\, dx_0$$

$$+ \frac{1}{2\pi i} \int_{\alpha_0}^{\alpha_1} \gamma(\alpha) \ln \frac{x_0 - \sqrt{x_0^2 - a^2} - z - \sqrt{z^2 - a^2}}{x_0 + \sqrt{x_0^2 - a^2} - z - \sqrt{z^2 - a^2}}\, d\alpha$$

$$+ \frac{\Gamma_0}{2\pi i} \ln(z + \sqrt{z^2 - a^2})^2 \qquad (3)$$

If the bottom has the form of a parabola, we may put

$$y = f(x) = -ex^2$$

whence

$$f'(x) = -2ex$$

In that case, we shall have

$$\frac{dw}{dz} = iv_1\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right) + \frac{i(\omega + 2ec)}{2}\,\frac{(z - \sqrt{z^2 - a^2})^2}{\sqrt{z^2 - a^2}}$$

$$+ \frac{1}{2\pi i}\,\frac{1}{\sqrt{z^2 - a^2}} \int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\sqrt{x_0^2 - a^2}}{x_0 - z}\, d\alpha + \frac{\Gamma_0}{2\pi i\sqrt{z^2 - a^2}} \qquad (2a)$$

$$w(z) = iv_1(z - \sqrt{z^2 - a^2}) - \frac{i(\omega + 2ec)}{4}(z - \sqrt{z^2 - a^2})^2 +$$

$$+ \frac{1}{2\pi i}\int_{\alpha_0}^{\alpha_1} \gamma(\alpha) \ln \frac{x_0 - \sqrt{x_0^2 - a^2} - z - \sqrt{z^2 - a^2}}{x_0 + \sqrt{x_0^2 - a^2} - z - \sqrt{z^2 - a^2}}\, d\alpha +$$

$$+ \frac{\Gamma_0}{2\pi i} \ln(z + \sqrt{z^2 - a^2}) \qquad (3a)$$

Comparing these formulas with formulas (2) and (3), we see that curving the bottom in the form of a parabola is analogous to the rotation of a flat plate. (For a flat plate we may put $f'(x_0) = 0$.)

The condition that the velocity be finite at point M requires that the coefficient of

$$\frac{1}{2\pi i \sqrt{z^2 - a^2}}$$

in the expression for dw/dz become zero when $z = -a$. Thus,

$$-2\int_{-a}^{+a} \frac{v_n \sqrt{a^2 - x_0^2}}{x_0 + a} dx_0 + \int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha) \sqrt{x_0^2 - a^2}}{x_0 + a} d\alpha_0 + \Gamma_0 = 0$$

or

$$\Gamma_0 - 2\int_{-a}^{+a} v_n \sqrt{\frac{a - x_0}{a + x_0}} dx_0 = \int_{\alpha_0}^{\alpha_1} \gamma(\alpha) \sqrt{\frac{x_0 - a}{x_0 + a}} d\alpha$$

We shall set

$$\frac{1}{\pi}\int_{-1}^{+1} f'(au) \sqrt{\frac{1 - u}{1 + u}}\, du = \beta_1 \qquad (4)$$

so that if

$$f'(x) = \sum_{n=2}^{\infty} nA_n \left(\frac{x}{a}\right)^{n-1}$$

then

$$\beta_1 = \sum_{k=1}^{\infty} [(2k + 1)A_{2k+1} - 2kA_{2k}]\frac{(2k - 1)(2k - 3)\ldots 5\cdot 3\cdot 1}{2^k \cdot k!}$$

We then have

$$2\int_{-a}^{+a} v_n \sqrt{\frac{a - x_0}{a + x_0}}\, dx_0 = 2\pi a\left(v_1 + \frac{\omega a}{2} + c\beta_1\right) = 2\pi a v_2 \qquad (5)$$

For a flat plate $\beta_1 = 0$ and v_2 is the normal velocity of the plate at a point whose distance is one-fourth the wetted length from the trailing edge.

The condition that the velocity at point M be finite in the notation of (5) assumes the form

$$\Gamma_0 - 2\pi a v_2 = \int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\sqrt{\frac{2a + \alpha_1 - \alpha}{\alpha_1 - \alpha}}\, d\alpha \qquad (6)$$

The above relation is an integral equation of the Volterra type of the first kind. For given functions $v_2(\alpha_1)$ and $a(\alpha_1)$, it is possible with the aid of (6) to determine $\gamma(\alpha)$. For a thin wing, the function $v_2(\alpha_1)$ completely determines the strength of the vortices springing from the trailing edge in an unsteady motion of translation with small angle of attack.

For a steady planing motion with constant velocity c_0, we have $\omega = \gamma = 0$; $a = a_0 = \text{const.}$

Equation (6) in this case gives:

$$\Gamma_0 = 2\pi a_0 v_{20} = 2\pi a_0(\beta_0 + \beta_1)\, c_0$$

Considering the unsteady motion as a disturbance from the state of steady motion, we may write equation (6) thus:

$$-2\pi\Delta(av_2) = \int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\sqrt{\frac{2a + \alpha_1 - \alpha}{\alpha_1 - \alpha}}\, d\alpha \qquad (7)$$

where

$$\Delta(av_2) = av_2 - a_0 v_{20}$$

For a rigid wing profile particularly, α and β_1 are independent of time. If, in addition, the horizontal velocity at the trailing edge is constant: c = const., then

$$-2\pi\Delta(av_2) = 2\pi a\left[v + \frac{\omega a}{2} - c(\beta - \beta_0)\right]$$

Hence in this case the intensity of the trailing vortices does not depend on the wing profile.

At $\alpha = \alpha_1$ the kernel of equation (7) becomes infinite. To remove this difficulty, we shall multiply both sides of (7) by $\dfrac{1}{\sqrt{u - \alpha_1}}$ and integrate with respect to α_1 from α_0 to u.

We then obtain

$$\Phi(u) \equiv -2\pi\int_{\alpha_0}^{u} \frac{\Delta(av_2)d\alpha_1}{\sqrt{u - \alpha_1}} = \int_{\alpha_0}^{u} d\alpha_1\int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\sqrt{\frac{2a + \alpha_1 - \alpha}{(u - \alpha_1)(\alpha_1 - \alpha)}}\, d\alpha$$

Changing the order of integration according to the Dirichlet formula, we find

$$\Phi(u) = \int_{\alpha_0}^{u} \gamma(\alpha)K(u,\alpha)d\alpha \tag{8}$$

where

$$K(u,\alpha) = \int_{\alpha}^{u} \sqrt{\frac{2a + \alpha_1 - \alpha}{(u - \alpha_1)(\alpha_1 - \alpha)}}\, d\alpha_1 =$$

$$= \int_{-\pi/2}^{+\pi/2} \sqrt{2a + \frac{u - \alpha}{2}(1 - \sin\psi)}\, d\psi$$

(A change in the variables has been made here

$$\alpha_1 = \frac{u + \alpha}{2} - \frac{u - \alpha}{2}\sin\psi\ )$$

The kernel $K(u,\alpha)$ is finite and, if a is constant, depends only on $u - \alpha$ and may be expressed by elliptic functions. The integral equation (8) may be solved by the method of successive approximations.

For a motion for which the changes in a and v_2 may be considered as infinitesimal, we obtain from equation (7), limiting ourselves to values of the first order of smallness,

$$-2\pi\delta(av_2) = \int_{\alpha_0}^{\alpha_1} \gamma(\alpha) \sqrt{\frac{2a_0 + \alpha_1 - \alpha}{\alpha_1 - \alpha}}\, d\alpha \qquad (9)$$

For a constant value of a, equation (9) coincides with equation (7). Denoting

$$\int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\, d\alpha = -[\Gamma(\alpha_1) - \Gamma_0] \equiv -\Gamma'(\alpha_1)$$

and setting $a_0 + \alpha_1 - \alpha = a_0 s$, we may reduce equation (9) to the form:

$$2\pi\delta(av_2) - \Gamma'(\alpha_1) = a_0 \int_{1+\frac{\alpha_1-\alpha_0}{a_0}}^{1} \gamma(\alpha_1 + a_0 - a_0 s)\left[\sqrt{\frac{s+1}{s-1}} - 1\right] ds \quad (10)$$

If, after a sufficiently large time interval, the oscillations tend to approach a certain steady oscillating state, we may limit ourselves to a study of this state. For steady oscillations we may put $\alpha_0 = -\infty$. Equation (10) then assumes the form

$$2\pi\delta(av_2) - \Gamma'(a_1) = a_0 \int_{\infty}^{1} \gamma(\alpha_1 + a_0 - as)\left[\sqrt{\frac{s+1}{s-1}} - 1\right] ds \quad (11)$$

In equation (10) we may in general consider $\alpha_0 = -\infty$, if $\delta(av_2)$ is such that for $\alpha < \alpha_0$; $\delta(av_2) = 0$. Under this condition equation (11) holds true for an arbitrary motion.

IV. THE PROBLEM OF LANDING ON THE STEP

Side by side with the problem of steady planing, it is not difficult to obtain a complete solution of the problem of the landing with constant velocity of a flat step on undisturbed water. The solution of this problem is rendered simple by the fact that during the immersion with constant velocity in a weightless liquid of a wedge, in the case of the two-dimensional problem, or of a coneshaped body. In the case of the three-dimensional problem, the motions of the fluid at different time instants are dynamically similar. The problem of the landing on a step with constant velocity has been considered in the work of Wagner (reference 1). We shall consider the problem once more here and add a few simple results.

Let us consider a step which settles with constant velocity c on an undisturbed water surface. We shall denote by β (fig. 3) the angle the bottom makes with the horizontal and by κ the angle between the horizontal and the landing velocity. We shall assume the angles β and κ both to be very small.

Since the motions of the fluid at various instants are dynamically similar, the wetted length $2a$ and the circulation along it are proportional to time. Hence

$$\gamma = -\frac{1}{c}\frac{d\Gamma}{dt} = \text{const}$$

and therefore equation (7), section III, gives

$$-2\pi a v_2 = \gamma \int_0^{\alpha_1} \sqrt{\frac{2a + \alpha_1 - \alpha}{\alpha_1 - \alpha}}\, d\alpha \tag{1}$$

Setting $\alpha = au$, $\frac{\alpha_1 + a}{a} = s_0 = \text{const} > 1$, and replacing v_2 by $c(\beta + \kappa)$, we find

$$-2\pi c(\beta + \kappa) = \gamma \int_0^{s_0 - 1} \sqrt{\frac{s_0 + 1 - u}{s_0 - 1 - u}}\, du$$

whence

$$\gamma = -\frac{2\pi c(\beta + \kappa)}{\sqrt{s_0^2 - 1} + \ln(s_0 + \sqrt{s_0^2 - 1})} \tag{2}$$

The minus sign shows that the horizontal velocity behind the step on the surface of the water is directed toward the left.

With the aid of formula (2), section III, it is easy to determine the vertical component of the velocity of the fluid particles on the x - axis. Setting $x/a = s$, we obtain after a simple computation

$$v = c(\beta + \kappa)\left[\frac{1}{\sqrt{s_0^2-1} + \ln(s_0 + \sqrt{s_0^2-1})}\left(\sqrt{s_0^2-1}\sqrt{\frac{s+1}{s-1}} - \ln\frac{ss_0 + 1 - \sqrt{s^2-1}\sqrt{s_0^2-1}}{s + s_0}\right) - 1\right] \quad (3)$$

Using this value for the vertical velocity, it is not difficult to compute the rise η of the water at any point with coordinate $\alpha^* = \text{const}$

We have:

$$\alpha^* = \alpha_1 + a + x$$

Since

$$\alpha_1 = ct; \quad a = \frac{\alpha_1}{s_0 - 1}$$

and

$$x = sa = \frac{sct}{s_0 - 1}$$

therefore

$$\alpha^* = \frac{s_0 + s}{s_0 - 1}\, ct$$

whence

$$t = \frac{\alpha^* (s_0 - 1)}{c(s_0 + s)}$$

$$dt = -\frac{\alpha^*(s_0 - 1)}{c(s_0 + s)^2}\, ds$$

Thus, for the rise η, we find the formula

$$\eta = \int_0^t v dt = - \frac{\alpha^*(s_0 - 1)}{c} \int_\infty^s \frac{v ds}{(s_0 + s)^2} \tag{4}$$

At the forward edge $x = a$, $s = 1$, and $\alpha^* = (s_0 + 1)a$, so that here

$$\eta = - \frac{a(s_0^2 - 1)}{c} \int_\infty^1 \frac{v ds}{(s_0 + s)^2} \tag{4a}$$

Substituting v from (3) and performing the integration, we find

$$\eta = a(\beta + \kappa) \frac{s_0 \ln(s_0 + \sqrt{s_0^2 - 1}) - \sqrt{s_0^2 - 1}}{\ln(s_0 + \sqrt{s_0^2 - 1}) + \sqrt{s_0^2 - 1}} \tag{5}$$

On the other hand, from geometric considerations it follows that at the forward edge

$$\eta = a[2\beta - (s_0 - 1)\kappa] \tag{6}$$

Equating (5) and (6), we obtain the equation for the determination of s_0 as a function of β/κ.

If $l' = 2a$ is the wetted length and l is the portion of the wetted length below the water level, then

$$\frac{l'}{l} = \frac{2a\beta}{(s_0 - 1)a\kappa} = \frac{2}{s_0 - 1} \frac{\beta}{\kappa} \tag{7}$$

On figure 4 is plotted the curve showing how l'/l varies with β/κ in accordance with formula (7). When $\beta/ \longrightarrow \infty$, we have $l'/l \longrightarrow \infty$. It is worth noting that in formula (7) c does not enter and therefore the ratio l'/l does not depend on the landing velocity.

V. SOLUTION OF THE INTEGRAL EQUATION FOR STEADY OSCILLATIONS

We shall now consider the solution of the integral equation

$$2\pi\delta(av_2) - \Gamma'(\alpha_1) = a_0 \int_\infty^1 \gamma(\alpha_1 + a_0 - a_0 s)\left[\sqrt{\frac{s + 1}{s - 1}} - 1\right] ds \tag{1}$$

for steady oscillations about a certain steady planing motion. Let

$$\delta(av_2) = \underline{R}Ae^{ikt} = A_0 \cos (kt + \epsilon)$$

where $A = A_0 e^{i\epsilon}$; k is the frequency of the oscillations.

We shall consider small oscillations, such that the amplitude A_0 and the amplitude of the variations in the horizontal velocity of the trailing edge are very small. We shall assume the following law of motion for the point M:

$$\alpha_1 = c_0 t + \overline{\alpha_1}$$

where $\overline{\alpha_1}$ is a function of time and assumes only small values. To an accuracy of the second order of smallness, we obtain

$$\delta(av_2) = \underline{R}Ae^{i\frac{k\alpha_1}{c_0}}$$

The solution of equation (1) is found by setting

$$\Gamma'(\alpha_1) = \underline{R}De^{i\frac{k\alpha_1}{c_0}}$$

whence

$$\gamma(\alpha) = -\frac{d\Gamma'}{d\alpha} = \underline{R}\left[-\frac{ik}{c_0} De^{\frac{ik\alpha}{c_0}}\right] \tag{2}$$

Denoting the abstract number ka_0/c_0 by μ, we may write

$$\gamma(\alpha_1 + a_0 - a_0 s) = -\underline{R}D\frac{i\mu}{a_0}e^{\frac{ik\alpha_1}{c_0}}e^{i\mu}e^{-i\mu s}$$

Substituting the expressions for $\Gamma'(\alpha_1)$ and $\gamma(\alpha_1 + a_0 - a_0 s)$ in equation (1), discarding the symbol $\underline{R}$ and dividing by the common factor $e^{\frac{ik\alpha_1}{c_0}}$, we obtain

$$2\pi A - D = -i\mu De^{i\mu}\int_{\infty}^{1} e^{-i\mu s}\left[\sqrt{\frac{s+1}{s-1}} - 1\right]ds \tag{3}$$

The integral on the right-hand side may be expressed by a Bessel function. For this purpose, we shall make use of the integral representation of the Hankel functions in the Poisson form (reference 9).

$$H_p^{(1)}(\lambda) = \chi_p \int_{i\infty}^{1} e^{i\lambda w} (1 - w^2)^{p-\frac{1}{2}} dw$$

where

$$\lambda = \mu + i\sigma; \quad \mu > 0; \quad \chi_0 = \frac{2}{\pi}; \quad \chi_1 = \frac{2\lambda}{\pi}$$

For the path of integration, we may take any curve situated in the first quadrant of the w plane. Replacing

w by $Re^{i\vartheta}$ and assuming that $\sigma > 0$, we have

$$\lim_{R \to \infty} \int_0^{\pi/2} e^{-R(\sigma\cos\vartheta + \mu\sin\vartheta) + iR(\mu\cos\vartheta - \sigma\sin\vartheta)} (1 - w^2)^{p-\frac{1}{2}} iw d\vartheta = 0$$

and therefore

$$H_p^{(1)}(\lambda) = \chi_p e^{\left(\frac{1}{2}-p\right)\pi i} \int_{\infty}^{1} e^{i\lambda s} (s^2 - 1)^{p-\frac{1}{2}} ds \qquad (4)$$

Using the above formula, we find

$$\int_{\infty}^{1} e^{i\lambda s} \left[\sqrt{\frac{s+1}{s-1}} - 1\right] ds = \frac{\pi}{2}\left[-iH_0^{(1)}(\lambda) + H_1^{(1)}(\lambda)\right] - \frac{e^{i\lambda}}{i\lambda} \qquad (5)$$

The above relation has been derived on the assumption that $\sigma > 0$ but it is obviously true also for $\sigma = 0$. Substituting $\sigma = 0$ and replacing i by $-i$, we find

$$\int_{\infty}^{1} e^{-i\mu s} \left[\sqrt{\frac{s+1}{s-1}} - 1\right] ds = \frac{\pi}{2}[iH_0^{(2)}(\mu) + H_1^{(2)}(\mu)] + \frac{e^{-i\mu}}{i\mu}$$

Substituting the value of the above integral in equation (3), we obtain

$$2\pi A = -\frac{\pi}{2} i\mu e^{i\mu} [iH_0^{(2)}(\mu) + H_1^{(2)}(\mu)] D$$

whence

$$D = \frac{4e^{-i\mu} A}{\mu [H_0^{(2)}(\mu) - iH_1^{(2)}(\mu)]} \qquad (6)$$

For large value of μ, replacing the Hankel functions by their asymptotic expressions, we find

$$D = A\sqrt{\frac{2\pi}{\mu}}\, e^{-\frac{\pi i}{4}}$$

Hence

$$\lim_{\mu \to \infty} \frac{D}{A} = 0$$

For small values of μ, we obtain

$$D = \frac{4\, Ae^{-i\mu}}{\frac{2}{\pi} + \mu + \frac{2i\mu}{\pi} \ln \frac{2}{\gamma\mu}}$$

where

$$\gamma = 1.781072$$

whence

$$\lim_{\mu \to 0} D = 2\pi A$$

This value of D may be found if the circulation about MN is determined in the same manner as in the case of steady planing and in the wing theory as in the theory of motion with constant circulation.

Since the integral equation is linear, it is possible with the aid of a Fourier series to obtain the solution for any periodic law of oscillation. If

$$\delta(av_2) = \underline{R} \sum_{n=1}^{\infty} A_n e^{\frac{ikn\alpha}{c_0}}$$

then

$$\Gamma'(\alpha) = \underline{R}\, \frac{4}{\mu} \sum_{n=1}^{\infty} \frac{A_n e^{-in\mu_0 \frac{ikn\alpha}{c_0}}}{n[H_0^{(2)}(n\mu) - iH_1^{(2)}(n\mu)]}$$

By an analogous method with the aid of a Fourier integral may be found the solution of integral equation (1) at an arbitrary value of $\delta(av_2)$, provided

$$\int_{-\infty}^{\alpha} |\delta(av_2)|\, d\alpha$$

remains finite in the region of α considered.

VI. HYDRODYNAMIC FORCES IN UNSTEADY PLANING OR UNSTEADY WING MOTION

We shall now establish general formulas for the hydrodynamic forces with the aid of the methods given above for the determination of the flow. For simplicity, we shall limit our study to the case where the bottom is of parabolic form.

Let X and Y denote respectively the horizontal and vertical component and M the moment about the center of the wetted length of the hydrodynamic force acting on a unit width of the planing surface. The pressure on the bottom we shall determine with the aid of the Lagrange integral. Neglecting the squares of the value of the absolute velocity of the fluid, we may write

$$Y = \int_{-a}^{+a} (p - p_o)dx = -\rho \int_{-a}^{+a} \frac{\partial\varphi(\alpha,t)}{\partial t}\, dx \qquad (1)$$

$$X = -\int_{-a}^{+a} (p - p_o)\frac{dy}{dx}dx = -\beta Y + \rho \int_{-a}^{+a} f'(x)\frac{\partial\varphi(\alpha,t)}{\partial t}\, dx \qquad (2)$$

$$M = \int_{a}^{+a} (p - p_o)x dx = -\rho \int_{-a}^{+a} x\frac{\partial\varphi(\alpha,t)}{\partial t}\, dx \qquad (3)$$

In the case under consideration, $f'(x) = -2ex$, so that

$$X = -\beta\, Y + 2e\, M \qquad (I)$$

It remains to compute Y and M. The variables α and x are connected by the relation

$$\alpha = x + \int_{t_o}^{t} \left(c + \frac{da}{dt}\right) dt + \text{const}$$

and therefore

$$\frac{\partial\varphi(\alpha,t)}{\partial t} = \frac{\partial\varphi(x,t)}{\partial t} - \frac{\partial\varphi(x,t)}{\partial x}\left(c + \frac{da}{dt}\right) \qquad (4)$$

Noting that

$$\varphi(+a) - \varphi(-a) = \frac{\Gamma(\alpha_1)}{2} = \frac{\Gamma_0 + \Gamma'}{2}$$

where

$$\Gamma' = -\int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\,d\alpha$$

we obtain

$$Y = \frac{1}{2}\rho\left(c + \frac{da}{dt}\right)\Gamma_0 - \frac{1}{2}\rho\left(c + \frac{da}{dt}\right)\int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\,d\alpha - \rho\int_{-a}^{+a} \frac{\partial\varphi(x,t)}{\partial t}dx \quad (5)$$

The value of $\frac{\partial\varphi(x,t)}{\partial t}$ may be easily found, using formula (3a), section III:

$$\frac{\partial\varphi(x,t)}{\partial t} = \underline{R}\,\frac{\partial w(z,t)}{\partial t} = -\frac{dv_1}{dt}\sqrt{a^2 - x^2} + \frac{d\omega_1}{dt}\,\frac{1}{2}\,x\sqrt{a^2 - x^2}$$

$$+ \frac{1}{2\pi}\left(c + \frac{da}{dt}\right)\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\sqrt{a^2 - x^2}\,d\alpha}{(x - x_0)\sqrt{x_0^2 - a^2}} - \frac{da}{dt}\,\frac{\Gamma_0\,x}{2\pi a\sqrt{a^2 - x^2}}$$

$$- \frac{da}{dt}\left[\frac{av_1}{\sqrt{a^2 - x^2}} - \frac{\omega_1\,a\,x}{2\sqrt{a^2 - x^2}} + \frac{1}{2\pi a}\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)(xx_0 + a^2)\,d\alpha}{\sqrt{a^2 - x^2}\sqrt{x_0^2 - a^2}}\right] \quad (6)$$

where ω_1 denotes $\omega + 2ec$.

Substituting now $\frac{\partial\varphi}{\partial t}$ from (6) in formula (5) and noting that

$$\int_{-a}^{+a} \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{2}; \qquad \int_{-a}^{+a} \frac{dx}{\sqrt{a^2 - x^2}} = \pi;$$

$$\int_{-a}^{+a} \frac{\sqrt{a^2 - x^2}}{x - x_0}\,dx = -\pi(x_0 + \sqrt{x_0^2 - a^2});$$

$$\int_{-a}^{+a} x\sqrt{a^2 - x^2}\,dx = \int_{-a}^{+a} \frac{xdx}{\sqrt{a^2 - x^2}} = 0$$

we find

$$Y = \frac{1}{2}\rho\left(c + \frac{da}{dt}\right)\Gamma_0 + \frac{\rho\pi a^2}{2}\frac{dv_1}{dt} + \rho\pi a\, v_1 \frac{da}{dt}$$

$$+ \rho\left(c + \frac{da}{dt}\right)\frac{1}{2}\int_{\alpha_0}^{\alpha_1} \frac{x_0\gamma(\alpha)d\alpha}{\sqrt{x_0^2 - a^2}} + \frac{\rho a}{2}\frac{da}{dt}\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}} \quad (7)$$

To simplify this formula, we shall make use of integral equation (6), section III, from which, noting that

$$|x_0| > a, \quad x_0 < 0$$

we have

$$\int_{\alpha_0}^{\alpha_1} \frac{x_0\gamma(\alpha)d\alpha}{\sqrt{x^2 - a^2}} = -\int_{\alpha_0}^{\alpha_1} \gamma(\alpha)\sqrt{\frac{x_0 - a}{x_0 + a}}\,d\alpha + a\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}}$$

$$= 2\pi a\, v_2 - \Gamma_0 + a\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}}$$

Using the above equation, we obtain for the lift force the final formula

$$Y = \frac{d}{dt}\left(\frac{\rho\pi a^2}{2} v_1\right) + \rho\pi a\left(c + \frac{da}{dt}\right)v_2$$

$$+ \frac{\rho a}{2}\left(c + 2\frac{da}{dt}\right)\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}} \quad (II)$$

For steady planing

$$\gamma = 0; \quad a = \text{const}; \quad v_1 = v_2 = c\beta = \text{const}$$

and in this case, formula (II) gives

$$Y_0 = \rho\pi ac\, v_2 = \rho\pi ac^2 \beta \text{ *}$$

This is the well-known formula of Wagner for the lift force in planing.

In studying the motion of the fluid during the immersion of a slightly inclined V bottom, Wagner identifies each of its instantaneous conditions of motion with that due to the impact of a plate of width 2a which, after impact, has received a velocity equal to that of the sinking bottom at the instant of time considered. He thus obtains for the lift the formula

$$Y_1 = \frac{d}{dt}\left(\frac{\rho\pi a^2}{2} v_1\right)$$

This assumption corresponds in our considerations to the substitution in formula (7) of the values $c = 0$; $\gamma = 0$; $\Gamma_0 = 0$; after which it agrees with the Wagner formula. We may note that this is the first term on the right-hand side of formula (II). This term takes account of the effect of the so-called added mass. The term $\rho\pi a\left(c + \frac{da}{dt}\right)v_2$ gives the lift force, which may be obtained on the "stationary hypothesis" and finally the last term takes into account the effect of the horizontal velocities of the fluid on the free surface, which velocities are absent in steady planing.

We shall now compute the magnitude of the moment. On the basis of (3) and (4), we have

$$M = \rho\int_{-a}^{+a}\left[x\frac{\partial\varphi(x,t)}{\partial x}\left(c + \frac{da}{dt}\right) - x\frac{\partial\varphi(x,t)}{\partial t}\right]dx \tag{8}$$

From formula (2a), section III, we find

$$x\frac{\partial\varphi}{\partial x} = v_1\frac{x^2}{\sqrt{a^2 - x^2}} + \frac{\omega_1}{2}\frac{(a^2 - 2x^2)x}{\sqrt{a^2 - x^2}} + \frac{\Gamma_0 x}{2\pi\sqrt{a^2 - x^2}}$$

$$+ \frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1}\frac{\gamma(\alpha)x\sqrt{x_0^2 - a^2}}{(x_0 - x)\sqrt{a^2 - x^2}}\,d\alpha \tag{9}$$

---

*Independently of Wagner and on the basis of other considerations, an analogous formula was established by N. A. Sokolov (reference 10) for the dynamic component of the lift in the case of steady planing.

Substituting in (8) the value of $x\frac{\partial\varphi}{\partial x}$ from (9) and the value of $\frac{\partial\varphi(x,t)}{\partial t}$ from (7), and noting that

$$\int_{-a}^{+a} x^2\sqrt{a^2 - x^2}\,dx = \frac{\pi a^4}{8}; \quad \int_{-a}^{+a} \frac{x^2 dx}{\sqrt{a^2 - x^2}} = \frac{\pi a^2}{2}$$

we obtain

$$M = -\frac{\rho\,\pi a^4}{16}\frac{d\omega_1}{dt} - \frac{\rho\,\pi a^3}{4}\omega_1\frac{da}{dt} + \frac{\rho\,\pi a^2}{2}v_1\left(c + \frac{da}{dt}\right) + \frac{\rho a}{4}\frac{da}{dt}\Gamma_0$$

$$-\frac{\rho a}{4}\frac{da}{dt}\int_{\alpha_0}^{\alpha_1}\gamma(\alpha)\sqrt{\frac{x_0 - a}{x_0 + a}}\,d\alpha + \frac{\rho a^2}{4}\left(c + 2\frac{da}{dt}\right)\int_{\alpha_0}^{\alpha_1}\frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}} \tag{10}$$

Since

$$\int_{\alpha_0}^{\alpha_1}\gamma(\alpha)\sqrt{\frac{x_0 - a}{x_0 + a}}\,d\alpha = \Gamma_0 - 2\pi a v_2 = \Gamma_0 - 2\pi a v_1 - \pi a^2\omega_1$$

formula (10) is considerably simplified to

$$M = -\frac{\rho\pi a^4}{16}\frac{d\omega_1}{dt} + \frac{\rho\pi a^2}{2}v_1\left(c + 2\frac{da}{dt}\right) + \frac{\rho a^2}{4}\left(c + 2\frac{da}{dt}\right)\int_{\alpha_0}^{\alpha_1}\frac{\gamma(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}} \tag{III}$$

For a flat plate, ω_1 is simply the angular rotational velocity ω and for a plate bent into the form of an arc of a parabola $y = -ex^2$,

$$\omega_1 = \omega + 2ec$$

Formulas (II) and (III) were derived for the case of planing. By simply doubling the right-hand sides, we obtain the formulas for the lift and moment of a thin wing with variable chord in unsteady or non-uniform motion with separation of a line of discontinuity of velocity.

The values of φ and $\frac{\partial\varphi}{\partial t}$ above and below are, in fact, equal in magnitude but opposite in sign. Hence, the increase in pressure below and the decrease in pressure above, due to $\frac{\partial\varphi}{\partial t}$, are exactly equal. For this reason, to obtain the force acting on the wing, it is necessary to double the forces obtained for the planing motion. We may note further that for the case of a wing the term containing v^2 in the formula for the pressure is not necessarily neglected, since the value of the absolute velocity of the fluid above and below are exactly equal to each other, so that this term does not represent a pressure jump in passing through MN. The term with v^2 in the formula for the pressure on a wing is equivalent to the appearance of suction forces directed to the right at the sharp leading edge. This concentrated force modifies the hydrodynamic action of the fluid as a result of the rarefaction which actually takes place at the rounded leading edge of a wing. To obtain the horizontal component of the hydrodynamic force acting on the wing, it is necessary to double the horizontal component that acts in planing and add to it the suction force.

We shall denote by P the suction force at the leading edge of a wing. The value of P is readily computed since

$$P = \frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz$$

The integration is performed in the counterclockwise direction about an infinitely small circle C with center at point $z = a$. At this point $\left(\frac{dw}{dz}\right)^2$ has a simple pole. Excluding the pole $z = a$, we obtain

$$P = -\rho\pi \lim_{z \to a} (z - a) \left(\frac{dw}{dz}\right)^2$$

Using the expression for $\frac{dw}{dz}$ from (2), section III, we find

$$P = \frac{\rho\pi}{2a}\left[v_1 a - \frac{\omega a^2}{2} + \frac{c}{\pi}\int_{-a}^{+a} \frac{f'(x_0)\sqrt{a^2 - x_0^2}}{a - x_0} dx_0 \right.$$
$$\left. + \frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\sqrt{x_0^2 - a^2}}{x_0 - a} d\alpha + \frac{\Gamma_0}{2\pi}\right]^2$$

On the basis of equation (6), section III, in the notation of (5), (4), section III, we have

$$\frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1}\frac{\bar{\gamma}(\alpha)\sqrt{x_0^2-a^2}}{x_0-a}\,d\alpha = \frac{a}{\pi}\int_{\alpha_0}^{\alpha_1}\frac{\bar{\gamma}(\alpha)\,d\alpha}{\sqrt{x_0^2-a^2}}$$

$$-\frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1}\bar{\gamma}(\alpha)\sqrt{\frac{x_0-a}{x_0+a}}\,d\alpha$$

$$= \frac{a}{\pi}\int_{\alpha_0}^{\alpha_1}\frac{\bar{\gamma}(\alpha)\,d\alpha}{\sqrt{x_0^2-a^2}} - \frac{\Gamma_0}{2\pi} + a\left(v_1 + \frac{\omega a}{2} + c\beta_1\right)$$

and

$$\frac{c}{\pi}\int_{-a}^{+a}\frac{f'(x_0)\sqrt{a^2-x_0^2}}{a-x_0}\,dx_0 = \frac{2ac}{\pi}\int_{-a}^{+a}\frac{f'(x_0)dx_0}{\sqrt{a^2-x_0^2}}$$

$$-\frac{c}{\pi}\int_{a}^{+a}f'(x_0)\sqrt{\frac{a-x_0}{a+x_0}}\,dx_0 = \frac{2ac}{\pi}\int_{-a}^{+a}\frac{f'(x_0)dx_0}{\sqrt{a^2-x_0^2}} - ac\beta_1$$

Therefore:

$$P = 2\rho\pi a\left[v_1 + \frac{c}{\pi}\int_{-a}^{+a}\frac{f'(x_0)dx_0}{\sqrt{a^2-x_0^2}} + \frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1}\frac{\bar{\gamma}(\alpha)\,d\alpha}{\sqrt{x_0^2-a^2}}\right]^2 \qquad \text{(IV)}$$

particularly for a motion with constant circulation

$$P = 2\rho\pi a\left[v_1 + \frac{c}{\pi}\int_{-a}^{+a}\frac{f'(x_0)dx_0}{\sqrt{a^2-x_0^2}}\right]^2$$

We shall bring out more in detail the significance of each of the components of the lift determined by formula (II).

For simplicity, we shall consider the plate as flat. We then have for the unsteady planing

$$\left.\begin{aligned}
Y_1 &= \frac{d}{dt}\left(\frac{\rho\pi a^2}{2} v_1\right) \\
Y_2 &= \rho\pi a \left(c + \frac{da}{dt}\right) v_2 \\
Y_3 &= \frac{\rho a}{2}\left(c + 2\frac{da}{dt}\right) \int_{\alpha_0}^{\alpha_1} \frac{\gamma(\alpha)\, d\alpha}{\sqrt{x_0^2 - a^2}} \\
M_1 &= -\frac{\rho\pi a^4}{16}\frac{d\omega}{dt} + \frac{\rho\pi a^2}{2} v_1\left(c + 2\frac{da}{dt}\right)
\end{aligned}\right\} \qquad (V)$$

and for the unsteady motion of a wing with $a = \text{const}$

$$\left.\begin{aligned}
\overline{Y}_1 &= \rho\pi a^2 \frac{dv_1}{dt} \\
\overline{Y}_2 &= 2\rho\pi a c v_2 \\
\overline{Y}_3 &= \rho a c \int_{\alpha_0}^{\alpha_1} \frac{\overline{\gamma}(\alpha)\, d\alpha}{\sqrt{x_0^2 - a^2}} \\
\overline{M}_1 &= -\frac{\rho\pi a^4}{8}\frac{d\omega}{dt} + \rho\pi a^2 v_1 c \\
P &= 2\rho\pi a\left[v_1 + \frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1} \frac{\overline{\gamma}(\alpha)\, d\alpha}{\sqrt{x_0^2 - a^2}}\right]^2
\end{aligned}\right\} \qquad (Va)$$

The forces Y_1, Y_2, and Y_3 are perpendicular to the plate (fig. 5), Y_1 and Y_2 being applied at the center and Y_3 at a distance from the leading edge equal to one-fourth the wetted length.

If we consider the classical problem of the motion of a plate of variable width $2a$ within an infinite fluid moving as a potential and continuous flow everywhere except at the edge of the plate, then the general hydrodynamic reaction is reduced to suction forces at the edges, the force Y_1 and the moment

$$\overline{M'}_1 = 2\, M'_1 = -\frac{d}{dt}\left(\frac{\rho\pi a^4}{8}\omega\right) + \rho\pi a^2 v_1\left(c + \frac{da}{dt}\right)$$

which we obtain by setting in formula (10) $\Gamma_0 = \gamma = 0$ and doubling the right-hand side. The force Y_1 and the moment M'_1 appear as a result of the so-called additional virtual masses. For a constant width 2a

$$\overline{M'_1} = \overline{M}_1 = 2M_1$$

Thus, the force Y_1 and the moment $\overline{M'_1}$ do not depend on the circulation Γ_0 and on the vortices springing from the trailing edge.

If we consider a motion with constant circulation for which motion the condition of finiteness of the velocity at point M is satisfied, we obtain the component $\overline{Y}_1 + \overline{Y}_2$ and the moment $\overline{M}_1$ (reference 11).

The force Y_3 or $\overline{Y}_3$ depends on the strength and positions of the separating vortices. As the steady condition is approached

$$Y_1 \rightarrow 0, \quad Y_3 \rightarrow 0, \quad Y \rightarrow Y_2$$

The expression for $\overline{Y}_2$ for unsteady motion coincides with the expression for the entire component $\overline{Y}$ for steady motion with constant translational and angular velocities. We have, in fact,

$$\overline{Y}_2 = 2\pi\rho a c v_2 \tag{11}$$

and, on the other hand, for steady motion, if

$$v_2 = \text{const}; \quad \omega = \text{const}$$

we have (see work of L. I. Sedov (reference 11) pp. 30 and 31)

$$\overline{Y} = \rho c \Gamma$$

and

$$\Gamma = 2\pi a v_2$$

that is:

$$\overline{Y} = 2\pi\rho a c v_2 \tag{12}$$

The right-had sides of formulas (11) and (12) agree not only in outward form but in actual significance as well, if by c and v_2 in formula (12) we understand the variable velocities for the given unsteady motion. Thus, $\overline{Y}_2$ is that value which we should obtain for the entire component $\overline{Y}$ if we were to compute it for the unsteady motion according to the rules which hold for steady motion, i.e., if we assumed the stationary hypothesis.

For accelerated motion, if

$$\Delta(av_2) > 0$$

then

$$\gamma(\alpha) < 0$$

i.e., the water breaks away behind at the step. In this case, for $\left|2\frac{da}{dt}\right| < c$

$$Y_3 < 0$$

always. Hence, the lift force obtained on the stationary hypothesis with the virtual additional mass effect taken into account has a value too large. In the case of retarded motion, the reverse holds true. Here we are confronted with an inertia phenomenon for the lift forces.

The physical explanation given above for the forces in formula (V) may be useful in taking account of the finiteness of the span of a planing body. Thus, for example, the component Y_1 for bodies of various shapes may be obtained theoretically as well as experimentally.

VII. FORCES ACTING IN LANDING ON THE STEP

We shall determine the hydrodynamic forces acting on the step as it is immersed in the liquid with constant velocity. In the notation of section IV, we have:

$$\omega = 0; \quad v_1 = v_2 = c(\beta + \kappa)$$

and

$$\gamma = -\frac{2\pi c(\beta + \kappa)}{\sqrt{s_0^2 - 1} + \ln(s_0 + \sqrt{s_0^2 - 1})}$$

Formulas (V) now become

$$Y_1 = \frac{d}{dt}\left[\frac{\rho\pi a^2}{2}\, c(\beta + \kappa)\right]$$

$$Y_2 = \rho\pi a\left(c + \frac{da}{dt}\right) c(\beta + \kappa)$$

$$Y_3 = \rho\,\frac{a}{2}\left(c + 2\,\frac{da}{dt}\right)\gamma\int_0^{\alpha_1}\frac{d\alpha}{\sqrt{x_0^2 - a^2}}$$

$$M_1 = \frac{\rho\pi a^2}{2}\, c(\beta + \kappa)\left(c + 2\,\frac{da}{dt}\right)$$

Bearing in mind that

$$\frac{\alpha_1 + a}{a} = s_0; \quad \alpha_1 = ct; \quad a = \frac{ct}{s_0 - 1}; \quad \frac{da}{dt} = \frac{c}{s_0 - 1}$$

and

$$\int_0^{\alpha_1}\frac{d\alpha}{\sqrt{(\alpha - \alpha_1 - a)^2 - a^2}} = \ln\left(s_0 + \sqrt{s_0^2 - 1}\right)$$

we obtain

$$Y_1 = \rho\pi c^2 h\,\frac{\left(1 + \frac{\beta}{\kappa}\right)}{(s_0 - 1)^2} \tag{1}$$

$$Y_2 = \rho\pi c^2 h\,\frac{\left(1 + \frac{\beta}{\kappa}\right)}{(s_0 - 1)^2}\, s_0 \tag{2}$$

$$Y_3 = -\rho\pi c^2 h\,\frac{\left(1 + \frac{\beta}{\kappa}\right)}{(s_0 - 1)^2}\left[\frac{(s_0 + 1)\ln\left(s_0 + \sqrt{s_0^2 - 1}\right)}{\sqrt{s_0^2 - 1} + \ln\left(s_0 + \sqrt{s_0^2 - 1}\right)}\right] \tag{3}$$

where h is the immersion of the trailing edge, i.e.,

$$h = \kappa ct$$

Combining equations (1), (2), and (3), we find

$$Y = \rho\pi c^2 h\epsilon$$

where

$$\epsilon = \frac{\left(1 + \frac{\beta}{\kappa}\right) (s_0 + 1) \sqrt{s_0^2 - 1}}{(s_0 - 1)^2 [\sqrt{s_0^2 - 1} + \ln(s_0 + \sqrt{s_0^2 - 1})]} \tag{4}$$

We have further

$$M_1 = \frac{a}{2} (Y_1 + Y_2)$$

Thus, in landing on the step with constant velocity the general hydrodynamic reaction reduces itself to a single force applied at a distance from the leading edge equal to one-fourth the wetted length (fig. 6).

From formula (4), it is clear that this force depends essentially on the ratio β/κ. Figure 7 shows the nature of this dependence graphically. It is interesting to observe that the curve for ϵ has a minimum, a fact which indicates the existence of an optimum ratio β/κ in landing.

The impact force computed from formula (4) will actually be larger than the force applied at the step since we considered a flat bottom of infinite span. The finiteness of the span and the presence of a V angle result in a decreased water reaction.

In connection with the physical explanation of the hydrodynamic forces given in the preceding paragraph, it is interesting to compare with one another the values of Y_1, Y_2, and Y_3 for the case under consideration. Since all these forces are proportional to time, their ratios will be independent of time. For comparison, their variation with the parameter β/κ are shown graphically on figure 8, the values Y_1/k, Y_2/k, and Y_3/k, where

$$k = \frac{\rho\pi(\beta + \kappa)c^3 t}{(s_0 - 1)^2}$$

being laid off on the ordinate axis.

From formulas (1), (2), and (3),

$$\frac{Y_1}{k} = 1; \quad \frac{Y_2}{k} = s_0$$

$$\frac{Y_3}{k} = -\frac{(s_0 + 1)\ \ln(s_0 + \sqrt{s_0^2 - 1})}{\sqrt{s_0^2 - 1} + \ln(s_0 + \sqrt{s_0^2 - 1})}$$

We see that for small values of $\frac{\beta}{\kappa}$, Y_3 is comparable with Y_2 and in this case it is therefore incorrect to neglect to take into account the effect of the tangential velocity of the water behind the stop.

The limiting cases as $\frac{\beta}{\kappa} \rightarrow \infty$ and $\frac{\beta}{\kappa} \longrightarrow 0$ are shown on figures 9 and 10, respectively. In the first case, we have

$$\lim \frac{Y_2}{k} = \lim s_0 \sim 15.545; \qquad \lim \frac{Y_3}{k} \sim -2.995$$

in the second:

$$\lim \frac{Y_2}{k} = 1; \quad \lim \frac{Y_3}{k} = -1$$

VIII. FORCES ACTING DURING THE STEADY OSCILLATION OF A WING*

We shall consider the magnitude of the hydrodynamic forces during the steady harmonic oscillation of the wing. Wherever magnitudes appear in complex form, it is to be understood that only their real parts are considered.

Let the oscillations be of the form

$$\delta(av_2) = Ae^{ikt} \quad \text{and} \quad \delta(av_1) = Be^{ikt}$$

where

$$A = A_0 e^{i(\epsilon_1 + \epsilon_2)}; \qquad B = B_0 e^{i\epsilon_1}$$

A_0 and B_0 being small values. In the case of the wing, a = const.

Considering, for simplicity, c to be constant, we find from formulas (Va)

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*The problem of the steady oscillations of a plate has been treated by other methods in the paper by Glauert (reference 7) and by Keldysh and Lavrentiev (reference 8).

$$\overline{Y}_1 = \rho\pi a i k B e^{ikt} \tag{1}$$

$$\overline{Y}_2 - \overline{Y}_{20} = 2\rho\pi c A e^{ikt} \tag{2}$$

From formulas (2) and (6), section V, we have

$$\overline{\gamma}(\alpha) = -\frac{4iAe^{-i\mu}\, e^{\frac{ik\alpha}{c}}}{a[H_0^{(2)}(\mu) - iH_1^{(2)}(\mu)]} \tag{3}$$

Using the above and setting

$$x_0 = \alpha - \alpha_1 - a = -as$$

we find

$$\overline{Y}_3 = \rho ac \int_{-\infty}^{\alpha_1} \frac{\overline{\gamma}(\alpha)\,d\alpha}{\sqrt{x_0^2 - a^2}} = \frac{4\rho ciAe^{\frac{ik\alpha_1}{c}}}{[H_0^{(2)}(\mu) - iH_1^{(2)}(\mu)]} \int_{\infty}^{1} \frac{e^{-i\mu s}\,ds}{\sqrt{s^2 - 1}}$$

The integral on the right-hand side is expressed by Hankel functions of the second kind. From formulas (4), section V, at $p = 0$, and replacing i by $-i$, we obtain

$$\int_{\infty}^{1} \frac{e^{-i\mu s}\,ds}{\sqrt{s^2 - 1}} = \frac{\pi i}{2} H_0^{(2)}(\mu)$$

We shall furthermore denote

$$\frac{H_0^{(2)}(\mu)}{H_0^{(2)}(\mu) - iH_1^{(2)}(\mu)} = \nu(\mu)e^{i\chi(\mu)} \tag{4}$$

We then have finally

$$\overline{Y}_3 = -2\rho\pi cA\nu e^{i(kt+\chi)} \tag{5}$$

The magnitude ν as seen from formulas (2) and (5) is the ratio of the amplitude of the lift component, arising from the presence of the trailing vortices, to the increase in the lift force computed in accordance to the stationary hypothesis:

$$\nu = \frac{\max \overline{Y}_3}{\max(\overline{Y}_2 - \overline{Y}_{20})}$$

On figure 11 is shown how ν depends on $\mu = \frac{ka}{c}$. When $\mu = 0$, $\nu = 0$ and when $\mu = \infty$, $\nu = \frac{1}{2}$.

The angle χ is the phase shift of the force $-\overline{Y}_3$ with respect to the force $\overline{Y}_2 - \overline{Y}_{20}$. Figure 12 shows the dependence of χ on μ. When $\mu = 0$

$$\chi = \frac{\pi}{2}$$

and when $\mu = \infty$,

$$\chi = 0$$

The moment $\overline{M}_1$ in the case considered is now expressed as follows:

$$\overline{M}_1 - \overline{M}_{10} = -\frac{\rho\pi\ a^2}{4}(A - B)ike^{ikt} + \rho\pi\ acBe^{ikt} \tag{6}$$

At large frequency, i.e., when $\frac{ka}{c} > 1$, the effect of the trailing vortices results in a decrease in the lift force to one-half its value obtained on the stationary hypothesis.

$$\overline{Y}_3 \cong -\frac{1}{2}(\overline{Y}_2 - \overline{Y}_{20})$$

At small values of $\frac{ka}{c} \rightarrow 0$, i.e., at large horizontal velocities, the value of ν is near zero and therefore $\overline{Y}_3 \rightarrow 0$. In the latter case, the force obtained according to the theory of motion with constant circulation is near in value to the general actual force. It should be observed that, as may be seen from figure 12, $\overline{Y}_3$ approaches zero rather slowly when $\mu \rightarrow 0$.

The suction force P is determined with the aid of formulas (Va). Since

$$\frac{1}{2\pi}\int_{\alpha_0}^{\alpha_1}\frac{\overline{Y}(\alpha)d\alpha}{\sqrt{x_0^2 - a^2}} = \frac{\overline{Y}_3}{2\pi\rho\ ac} = -\underline{R}\,\frac{A}{a}\,\nu e^{i(kt+\chi)}$$

therefore

$$P = \frac{2\rho\pi}{a}\left[av_{10} + \delta(av_1) - \underline{R}A\nu\, e^{i(kt+\chi)}\right]^2$$

whence

$$P = \frac{2\rho\pi}{a}\left[av_{10} + B_0\cos(kt + \epsilon_1) - A_0\,\nu\cos(kt + \epsilon_1 + \epsilon_2 + \chi)\right]^2 \tag{7}$$

In conclusion, we shall consider the projection of the general hydrodynamic force on the horizontal direction. When the wing has the form of a flat plate, we have

$$\overline{X} = -(\overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3)\beta + P$$

where β is the small angle of inclination of the plate to the horizontal:

$$\beta = \beta_0 + \int \omega\, dt = \beta_0 + \frac{2}{a^2}\int (av_2 - av_1)dt$$

$$= \beta_0 + \frac{2}{a^2k} A_0 \sin(kt + \epsilon_1 + \epsilon_2) - \frac{2}{a^2k}\sin(kt + \epsilon_1)$$

The mean value of the horizontal component over a period is

$$\overline{X}_m = \frac{k}{2\pi}\int_0^{2\pi/k} \overline{X}\, dt = \frac{\rho\pi}{a}\left\{A_0^2\nu\left(\nu - 2\frac{c}{ak}\sin\chi\right) + A_0B_0\left[\cos\epsilon_2 - 2\nu\cos(\chi + \epsilon_2) + 2\frac{c}{ak}\left(\nu\sin(\chi + \epsilon_2) - \sin\epsilon_2\right)\right]\right\} \tag{8}$$

For translational oscillations

$$A_0 = B_0; \quad \epsilon_2 = 0$$

In this case:

$$\overline{X}_m' = \frac{\rho\pi A_0^2}{a}\left[1 + \nu^2 - 2\nu\cos\chi\right] > 0$$

i.e., we have a forward thrust on the wing.

For purely rotational oscillations about the center of the plate, $B_0 = 0$ and therefore

$$\overline{X}_m'' = \frac{\rho\pi A_0^2}{a}\nu\left[\nu - 2\frac{c}{ak}\sin\chi\right]$$

IX. FORCES DURING STEADY OSCILLATION OF A PLANING PLATE

In applying the general formulas derived in section VII, it is first necessary to determine a and $\frac{da}{dt}$ where $a = \frac{l'}{2}$ is half the wetted length. As tests have shown, the wetted length l' is always somewhat larger than the part l under the surface level. On figure 13 are presented data obtained from tests of Sottorf (reference 3) and those conducted at the Central Aero-Hydrodynamical Institute by Perelmuter (reference 12). The ratios l'/l are plotted as ordinates against the aspect ratios $\frac{l}{b} = \lambda$ as abscissas. The tests were carried out on flat plates. It is readily observed from consideration of the plotted data that $l' > l$. At large aspect ratios

$$l' \sim 1.1\ l$$

At small aspect ratios corresponding to a large span l' may exceed l considerably.

To determine theoretically the ratio l'/l for the case of steady planing motion of a plate of infinite span is not possible if we limit ourselves to the consideration of the motion of a weightless ideal fluid. Actually in the flow about an inclined plate the height h of the trailing edge above the level of the free surface (fig. 14) approaches infinity as the distance from the plate increases (reference 1). For this reason, it is also impossible from such considerations to determine the wetted length below the surface level. In the flow of a heavy fluid with spray formation, the fluid in front of the plate cannot rise higher than a certain level, depending on the planing speed. From the test data presented, it is seen that the wetted length depends essentially on the span of the planing plate.

Let us make the assumption that, in unsteady motion, l' depends only on l and that

$$l' = nl \tag{1}$$

where n is a constant coefficient. The value of n may be obtained from test data referring to that stationary state with respect to which we consider the disturbed motion to take place. Glauert and Perring in their paper on the stability of a seaplane in planing assume $l' = l$, i.e., $n = 1$ (reference 13).

It is necessary here to make the following reservation. On figure 13, it is seen that for small aspect ratios n depends very much on the aspect ratio. With unsteady planing, the aspect ratio is variable, so that for those cases where it is small we have no previous assurance that it is always permissible to neglect the variations in n. At large aspect ratios, however, the variations in n are negligible.

Considering n as constant and the disturbances small, δa may easily be obtained. We have

$$2\delta a = \delta l' = n\delta l = n\delta \frac{h}{\beta} = \frac{ny}{\beta_o} - \frac{2a_o\theta}{\beta_o}$$

where h is the immersion of the trailing edge, $y = \delta h$, $\theta = \delta\beta$.

$$\delta a = \frac{ny}{2\beta_o} - \frac{a_o\theta}{\beta_o} \tag{2}$$

$$\frac{da}{dt} = \frac{n}{2\beta_o}\frac{dy}{dt} - \frac{a_o}{\beta_o}\frac{d\theta}{dt} \tag{3}$$

We shall now proceed to the determination of the hydrodynamic forces. Retaining in formula II only small quantities of the first order, we have

$$Y - Y_o = \frac{\rho\pi\, a_o}{2}\frac{d(av_1)}{dt} + \rho\pi\, c_o\, \delta(av_2) + \rho\pi\, a_o c_o \beta_o\left(\delta\, c + \frac{3}{2}\frac{da}{dt}\right)$$

$$+ \rho\frac{a_o c_o}{2}\int_{\alpha_o}^{\alpha_1}\frac{\gamma(\alpha)d\alpha}{\sqrt{x_o^2 - a^2}} \tag{VI}$$

If M_2 is the moment of the hydrodynamic forces with respect to the trailing edge, then

$$M_2 = M + (a_o + \delta\, a)(Y - Y_o) + (a_o + \delta\, a)Y_o$$

From (III) and (VI), to an accuracy of the second order of smallness, we find

$$M_2 - M_{2\dot{o}} = -\frac{\rho\pi a_o^4}{16}\frac{d\omega}{dt} + \frac{\rho\pi a_o^2}{2}\frac{d(av_1)}{dt} + \frac{\rho\pi a_o c_o}{2}\delta(av_1) + \rho\pi a_o c_o \delta(av_2)$$

$$+\rho\pi a_o c_o^2 \beta_o\left(\frac{3}{2}\delta c + \frac{5}{2}\frac{da}{dt}\right) + \frac{3}{2}\rho\pi a_o c_o^2 \beta_o\ \delta a + \frac{3a_o}{2} Y_3 \qquad \text{(VII)}$$

The variations $\delta(av_1)$ and $\delta(av_2)$ may be expressed by the kinematic elements of motion of the trailing edge. We have

$$av_2 = (a_o + \delta a)(c_o + \delta c)(\beta_o + \theta) + (a_o + \delta a)\frac{dy}{dt} - \frac{(a_o + \delta a)^2}{2}\frac{d\theta}{dt}$$

and from formula (3), retaining only small quantities of the first order, we obtain

$$\delta(av_2) = \frac{c_o n}{2}\nu + a_o\frac{dy}{dt} - \frac{a_o^2}{2}\frac{d\theta}{dt} + a_o\beta_o\delta c \qquad (4)$$

and similarly

$$\delta(av_1) = \frac{c_o n}{2} y + a_o\frac{dy}{dt} - a_o^2\frac{d\theta}{dt} + a_o\beta_o\delta c \qquad (5)$$

In determining the forces, the greatest difficulty met with is in the computation of Y_3. To obtain this component, we shall consider small steady oscillations for which

$$\delta(av_2) = \underline{R}\ Ae^{ikt} \qquad (6)$$

The integral equation for the determination of $\Upsilon(\alpha)$ in this case is the same as for the wing and therefore

$$Y_3 = \frac{1}{2}\overline{Y}_3 = -\underline{R}\rho\pi Ac\nu e^{i(kt+X)} \qquad (7)$$

We shall now consider in detail the forces acting on the plate moving according to the law

$$\left.\begin{aligned} y &= \underline{R}\ Pe^{ikt} \\ \theta &= \underline{R}\ Qe^{ikt} \\ \delta c &= \underline{R}\ ik\ Ne^{ikt} \end{aligned}\right\} \qquad (8)$$

where P, Q, and N are complex constants.

Using relations (2), (3), (4), (5), and (8), we find from formulas (VI) and (VII):

$$Y - Y_0 = \underline{R}e^{ikt}\left\{\frac{\rho\pi c^2}{2} n(1 - \nu e^{i\chi})P - \rho\pi ac(1 + n - \nu e^{i\chi})ikP\right.$$

$$- \frac{\rho\pi a^2}{2} k^2 P - \rho\pi a^2 c\left(2 - \frac{\nu}{2} e^{i\chi}\right)ikQ + \frac{\rho\pi a^3}{2} k^2 Q$$

$$\left. + \rho\pi ca\beta(2 - \nu e^{i\chi}) ikN - \frac{\rho\pi a^2}{2} \beta k^2 N \right\} \qquad (9)$$

and

$$M_2 - M_{20} = \underline{R}e^{ikt}\left\{\frac{3}{2} \rho\pi ac^2 n\left(1 - \frac{1}{2}\nu e^{i\chi}\right)P\right.$$

$$+ \frac{3}{2} \rho\pi a^2 c(1 + n - \nu e^{i\chi})ikP - \frac{\rho\pi a^3}{2} k^2 P - \frac{3}{2} \rho\pi a^2 c^2 Q$$

$$- \frac{3}{2} \rho\pi a^3 c\left(2\frac{1}{3} - \frac{1}{2} \nu e^{i\chi}\right)ikQ + \frac{9}{16} \rho\pi a^4 k^2 Q$$

$$\left. + \frac{3}{2} \rho\pi a^2 c\beta(2 - \nu e^{i\chi}) ikN - \frac{\rho\pi a^3}{2} \beta k^2 N\right\} \qquad (10)$$

In the above formulas, the subscript o has been omitted from those magnitudes which correspond to steady planing.

Formulas (9) and (10) were derived by us from the general expressions for the hydrodynamic forces in unsteady planing. It is interesting to compare them with the results which may be obtained on the stationary hypothesis or on the assumptions of Glauert and Perring. The stationary hypothesis leads to the formulas following:

$$Y = Y_2 = \rho\pi a\left(c + \frac{da}{dt}\right) v_2$$

$$M_1 = \frac{\rho\pi a^2}{2}\left(c + 2\frac{da}{dt}\right) v_1$$

where M_1 is the moment with respect to the center of the wetted length. The moment with respect to the trailing edge is equal to

$$M_2 = M_1 + aY_2 = \frac{\rho\pi a^2}{2}\left(c + 2\frac{da}{dt}\right) v_1 + \rho\pi a^2 \left(c + \frac{da}{dt}\right) v_2$$

Retaining small terms of the first order only

$$Y - Y_0 = \rho\pi c\delta(av_2) + \rho\pi ac\beta\left(\delta c + \frac{da}{dt}\right)$$

$$M_2 - M_{20} = \frac{\rho\pi ac}{2}\,\delta(av_1) + \rho\pi ac\delta(av_2) + \frac{3}{2}\,\rho\pi ac^2\beta\delta a$$

$$+ \frac{3}{2}\,\rho\pi a^2 c\beta\,\left(\delta c + \frac{4}{3}\,\frac{da}{dt}\right)$$

The above formulas applied to the harmonic oscillations under consideration give

$$Y - Y_0 = \underline{R}\;e^{ikt}\left\{\frac{\rho\pi c^2 n}{2}\,P + \rho\pi ac\,\left(1 + \frac{n}{2}\right)ikP - \frac{3}{2}\,\rho\pi a^2 cikQ + 2\rho\pi ac\beta ikN\right\} \tag{11}$$

and

$$M_2 - M_{2\,0} = \underline{R}e^{ikt}\left\{\frac{3}{2}\,\rho\pi ac^2 nP + \frac{3}{2}\,\rho\pi a^2 c\,\left(1 + \frac{2n}{3}\right)ikP - \frac{3}{2}\,\rho\pi a^2 c^2 Q - 3\,\rho\pi a^3 cikQ + 3\rho\pi a^2 c\beta ikN\right\} \tag{12}$$

Glauert and Perring, in their paper cited above, make the following assumptions to obtain the hydrodynamic forces:

1) In unsteady planing motion, the lift force is computed according to the formula

$$Y = qsv^2\beta$$

where q is a constant coefficient proportional to the width of the plate and to the density of the fluid,

s the wetted length below the undisturbed water level (fig. 15),

v the magnitude of the velocity of the point of the plate coinciding with the center of pressure,

β the angle between the plate and the velocity direction.

2) In unsteady planing motion, the force Y is applied at a distance φs from the trailing edge where φ is the value in steady planing. For a plate $\varphi = 3/4$.

Using these assumptions, discarding terms of higher order of smallness, and setting $s = 2a$; $v_o = c$, we obtain

$$Y - Y_o = \frac{Y_o}{2a\beta} y + \frac{Y_o}{c\beta}\frac{dy}{dt} - \frac{3}{2}\frac{Y_o a}{c\beta}\frac{d\theta}{dt} + \frac{2Y_o}{c}\delta c$$

$$M_2 - M_{2o} = \frac{3Y_o}{2\beta} y + \frac{3Y_o a}{2c\beta}\frac{dy}{dt} - \frac{3Y_o a}{2\beta}\theta - \frac{9}{4}\frac{Y_o a}{c\beta}\frac{d\theta}{dt} + 3\frac{Y_o a}{c}\delta c$$

The values of the hydrodynamic forces obtained for the harmonic oscillations of a plate on the basis of the various assumptions made are compared in the table below. In transferring the data from formulas (9), (10), (11), and (12) to the table, the magnitude $\rho\pi ac^2\beta$ has been replaced by Y_o.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Wagner, Herbert: Über Stoss- und Gleitvorgänge an der Oberfläche von Flussigkieten. Z.f.a.M.M., vol. 12, no. 4, Aug. 1932, pp. 193-215.

2. Gurevitch, M. I., and Yanpolsky, A. P.: On the Motion of a Planing Plate. Technika Vozdushnovo Flota, no. 10, 1933.

3. Sottorf, Walter: Experiments with Planing Surfaces. T.M. No. 661, N.A.C.A., 1932, and T.M. No. 739, N.A.C.A., 1934.

4. Farren, W. S.: Proc., Third Int. Cong. f. App. Mech., Stockholm, 1930, I, p. 323.

5. Walker, P. B.: Growth of Circulation about a Wing and an Apparatus for Measuring Fluid Motion. R. & M. No. 1402, British A.R.C., 1931.

6. Wagner, H.: Über die Enstehung des dynamischen Auftriebes von Tragflügeln. Z.f.a.M.M., vol. 5, no. 1, Feb. 1925, pp. 17-35.

7. Glauert, H.: The Force and Moment on an Oscillating Aerofoil. R. & M. No. 1242, British A.R.C., 1929.

8. Keldysh, M. V., and Lavrentiev, M. A.: On the Theory of the Oscillating Wing. CAHI Tech. Note No. 45, 1935.

9. Jahnke und Ende: Funktionentafeln mit Formeln und Kurven. Teubner, 1928, p. 170.

10. Sokolov, N. A.: Hydrodynamic Properties of Planing Surfaces and Flying Boats. CAHI Trans. no. 149.

11. Chaplygin, S. A.: The Motion of a Cylinder Situated in a Plane Parallel Air Flow. CAHI Report No. 19, 1926.

Sedov, L. I.: On the Theory of the Unsteady Motion of an Airfoil in a Fluid. CAHI Report No. 229, 1935.

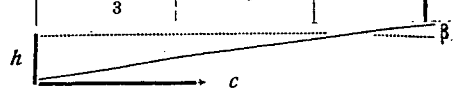
12. Perelmuter, A. S.: CAHI Tech. Note No. 48, 1935.

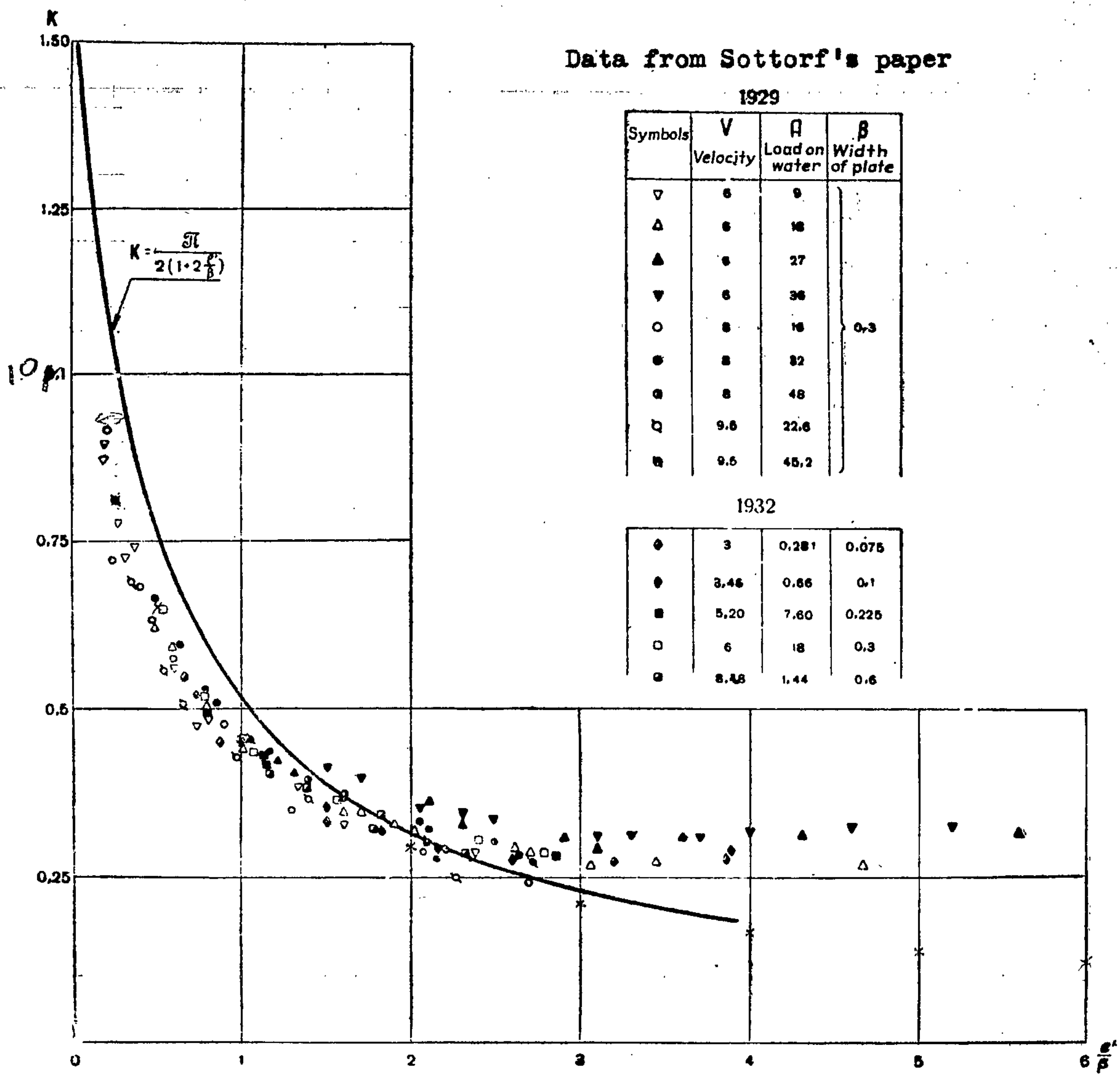
13. Perring, W. G. A., and Glauert, H.: The Stability on the Water of a Seaplane in the Planing Condition. R. & M. No. 1493, British A.R.C., 1933.

Comparison of Hydrodynamic Forces Obtained on the Basis of Various Assumptions for Steady Harmonic Oscillations with Respect to the Stationary Planing Condition

Coefficients	In the formulas for the lift $Y - Y_0$: General formula $\mu=\frac{ka}{c}$	Slow oscillations, large horizontal vel. $\mu\simeq 0$ $\nu=0$	Rapid oscillations $\mu>1$ $\nu=\frac{1}{2}$ $\chi=0$	General formula according to "stationary hypotheses"	According to assumptions of Glauert and Perring
$\frac{Y_0 Pe^{ikt}}{a}$	$(1-\nu e^{i\chi})\frac{n}{2\beta}$	$\frac{n}{2\beta}$	$\frac{n}{4\beta}$	$\frac{n}{2\beta}$	$\frac{1}{2\beta}$
	$(1+n-\nu e^{i\chi})\frac{i\mu}{\beta}$	$(1+n)\frac{i\mu}{\beta}$	$\left(n+\frac{1}{2}\right)\frac{i\mu}{\beta}$	$\left(\frac{n}{2}+1\right)\frac{i\mu}{2}$	$\frac{i\mu}{\beta}$
	$-\frac{\mu^2}{2\beta}$	$-\frac{\mu^2}{2\beta}$	$-\frac{\mu^2}{2\beta}$	0	0
$Y_0 Qe^{ikt}$	0	0	0	0	0
	$-(2-\frac{1}{2}\nu e^{i\chi})\frac{i\mu}{\beta}$	$-\frac{2i\mu}{\beta}$	$-1\frac{3}{4}\frac{i\mu}{\beta}$	$-1\frac{1}{2}\frac{i\mu}{\beta}$	$-1\frac{1}{2}\frac{i\mu}{\beta}$
	$\frac{\mu^2}{2\beta}$	$\frac{\mu^2}{2\beta}$	$\frac{\mu^2}{2\beta}$	0	0
$\frac{Y_0 Ne^{ikt}}{a}$	$(2-\nu e^{i\chi})\,i\mu$	$2i\mu$	$1\frac{1}{2}\,i\mu$	$2i\mu$	$2i\mu$
	$-\frac{\mu^2}{2}$	$-\frac{\mu^2}{2}$	$-\frac{\mu^2}{2}$	0	0
	In the formulas for the moment with respect to the trailing edge: $M_2 - M_{20}$				
$\frac{3}{2}\,Y_0 Pe^{ikt}$	$\left(1-\frac{1}{2}\nu n^{i\chi}\right)\frac{n}{\beta}$	$\frac{n}{\beta}$	$\frac{3}{4}\frac{n}{\beta}$	$\frac{n}{\beta}$	$\frac{1}{\beta}$
	$(1+n-\nu e^{i\chi})\frac{i\mu}{\beta}$	$(1+n)\frac{i\mu}{\beta}$	$\left(n+\frac{1}{2}\right)\frac{i\mu}{\beta}$	$\left(1+\frac{2}{3}n\right)\frac{i\mu}{\beta}$	$\frac{i\mu}{\beta}$
	$-\frac{\mu^2}{3\beta}$	$-\frac{\mu^2}{3\beta}$	$-\frac{\mu^2}{3\beta}$	0	0
$\frac{3}{2}\,Y_0 aQe^{ikt}$	$-\frac{1}{\beta}$	$-\frac{1}{\beta}$	$-\frac{1}{\beta}$	$-\frac{1}{\beta}$	$-\frac{1}{\beta}$
	$-\left(2\frac{1}{3}-\frac{\nu}{2}e^{i\chi}\right)\frac{i\mu}{\beta}$	$-2\frac{1}{3}\frac{i\mu}{\beta}$	$-2\frac{1}{2}\frac{i\mu}{\beta}$	$-2\frac{i\mu}{\beta}$	$-1\frac{1}{2}\frac{i\mu}{\beta}$
	$\frac{3}{8}\frac{\mu^2}{\beta}$	$\frac{3}{8}\frac{\mu^2}{\beta}$	$\frac{3}{8}\frac{\mu^2}{\beta}$	0	0
$\frac{3}{2}\,Y_0 Ne^{ikt}$	$(2-\nu e^{i\chi})\,i\mu$	$2i\mu$	$1\frac{1}{2}\,i\mu$	$2i\mu$	$2i\mu$
	$-\frac{\mu^2}{3}$	$-\frac{\mu^2}{3}$	$-\frac{\mu^2}{3}$	0	0

$\delta h = \Re Pe^{ikt};\ \delta\beta = \Re Qe^{ikt};\ \delta c = \Re ikNe^{ikt}$





Data from Sottorf's paper

1929

Symbols	V Velocity	A Load on water	β Width of plate
▽	6	9	0,3
△	6	18	
▲	6	27	
▼	6	36	
○	8	16	
●	8	32	
◐	8	48	
◔	9.6	22.6	
◕	9.6	45,2	

1932

◆	3	0,281	0.075
⬮	3,46	0.66	0.1
■	5,20	7.60	0.225
□	6	18	0.3
◙	8,48	1,44	0.6

Figure 1

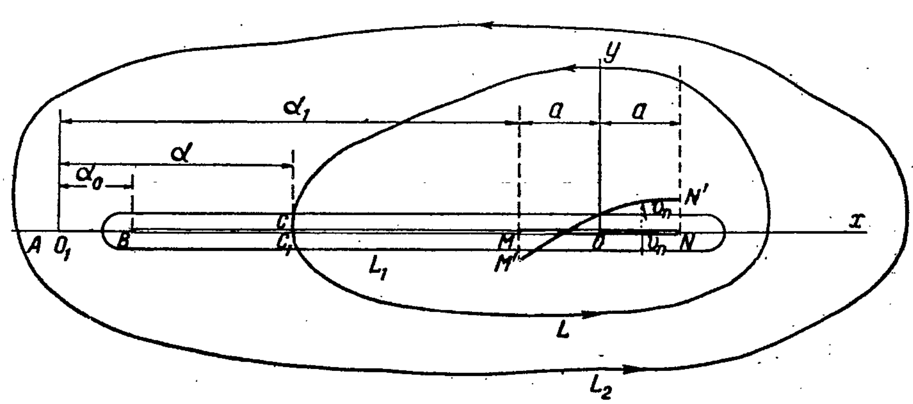
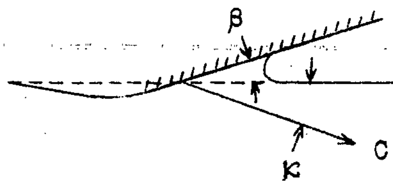


Figure 2



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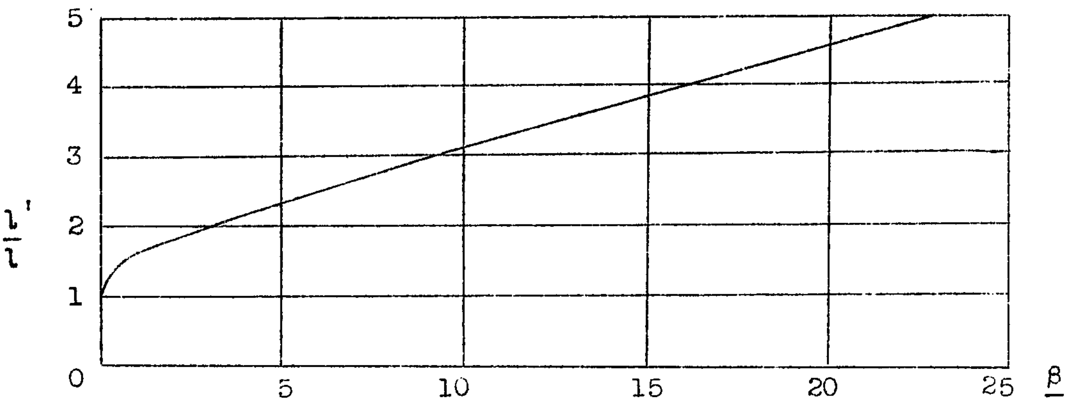

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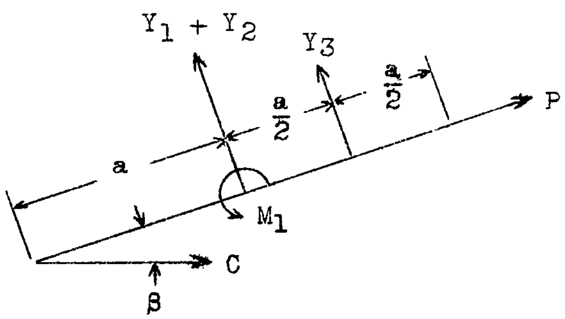

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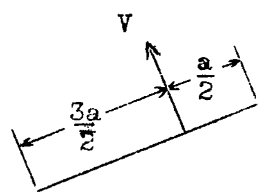

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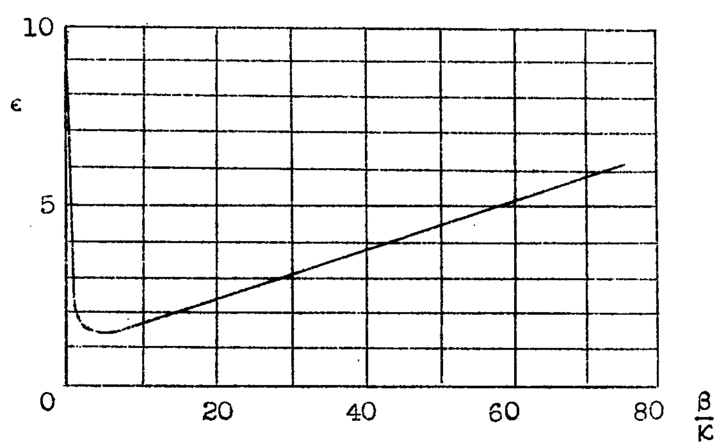

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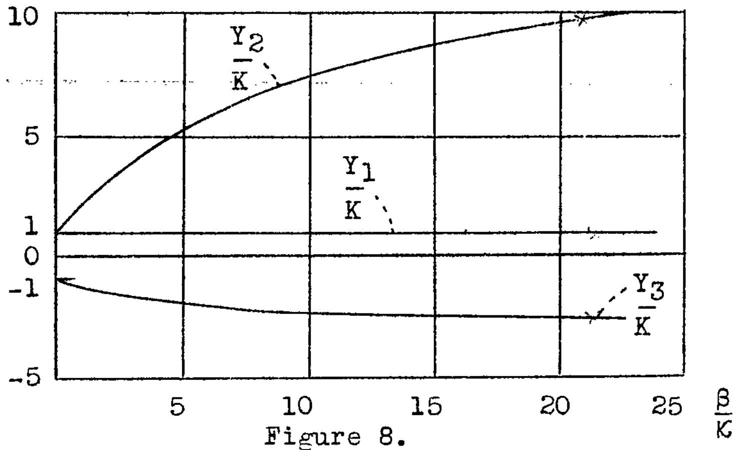

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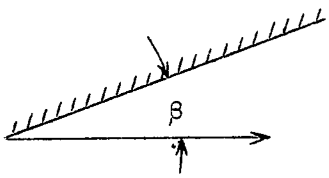

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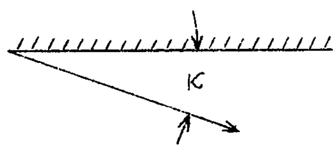

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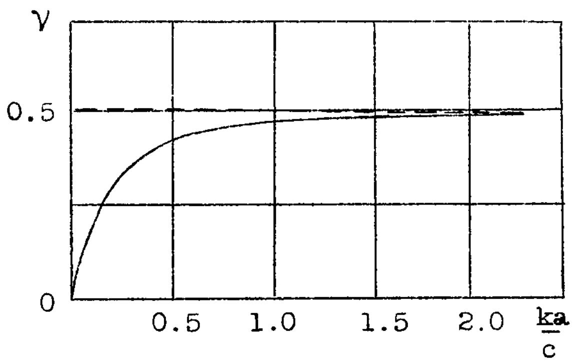
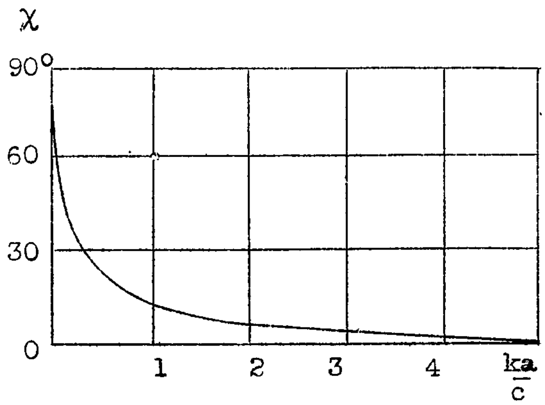


Figure 11.



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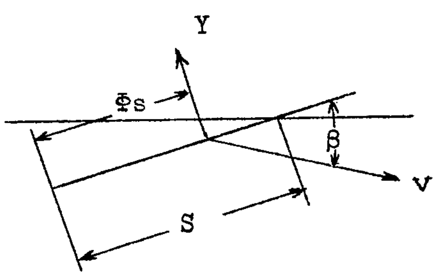

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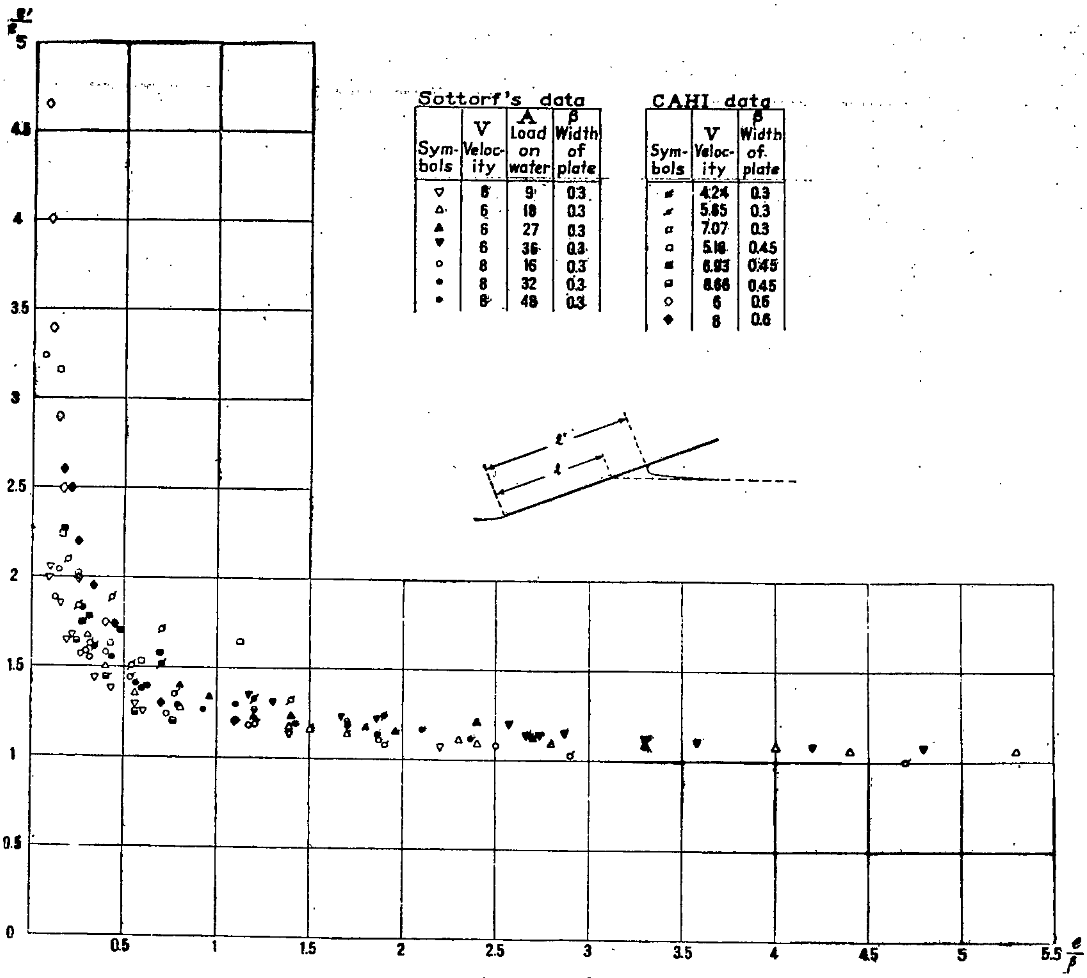

Sottorf's data

Symbols	V Velocity	A Load on water	β Width of plate
▽	6	9	0.3
△	6	18	0.3
▲	6	27	0.3
▼	6	36	0.3
○	8	16	0.3
●	8	32	0.3
●	8	48	0.3

CAHI data

Symbols	V Velocity	β Width of plate
[illegible]	4.24	0.3
[illegible]	5.65	0.3
[illegible]	7.07	0.3
□	5.18	0.45
■	6.93	0.45
[illegible]	8.66	0.45
◇	6	0.6
◆	8	0.6

Figure 13

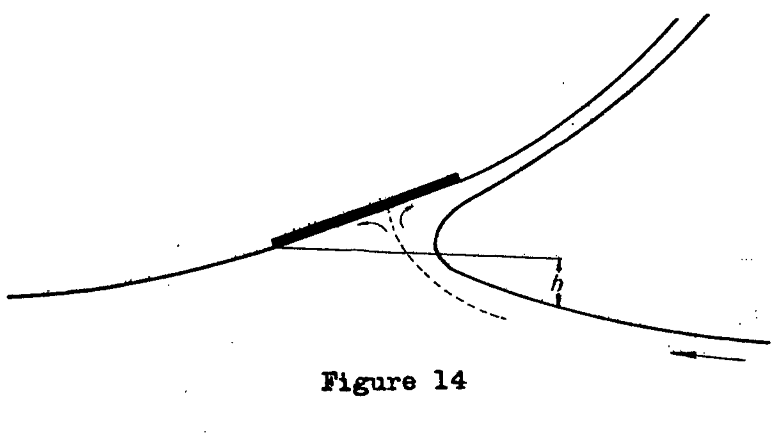

Figure 14